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ABILITIES OF FIRST-GRADE PUPILS TO LEARN MATERIALS IN TERMS OF ALGEBRAIC STRUCTURES BY TEACHING MACHINES.

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THE MAJOR PROBLEM OF THIS INVESTIGATION WAS TO DETERMINE TO WHAT EXTENT FIRST-GRADE PUPILS ARE CAPABLE OF LEARNING ALGEBRAIC STRUCTURES THROUGH PROGRAMED INSTRUCTION. IN THE EXPERIMENT APPROXIMATELY 130 FIRST-GRADERS WERE INSTRUCTED THROUGH AUDIOVISUAL TEACHING MACHINES FOR APPROXIMATELY 15 WEEKS. AT THE END OF THE PROGRAM, THE CHILDREN WERE TESTED THROUGH ORAL INTERVIEWS AND GROUP TESTS ON THEIR UNDERSTANDING OF BASIC MATHEMATICAL PRINCIPLES. THEIR PERFORMANCE WAS COMPARED WITH THAT OF TWO CONTROL GROUPS, WHO RECEIVED ONLY CONVENTIONAL INSTRUCTION. ONE CONTROL GROUP WAS DRAWN FROM THE FIRST GRADE, THE OTHER FROM THE FOURTH. IT WAS DISCOVERED THAT THE AUTOINSTRUCTIONAL PROGRAM WAS SUCCESSFUL IN DEVELOPING CERTAIN COMPETENCIES IN ARITHMETIC, SUCH AS ORDERING OF NUMBERS, WITH NO APPARENT LOSS IN ABILITY TO PERFORM SIMPLE ADDITION AND SUBTRACTION. HOWEVER, THE PRINCIPAL INVESTIGATION POINTS OUT THAT A CONCLUSION REGARDING THE MAJOR HYPOTHESIS, THE EXTENT TO WHICH FIRST-GRADERS ARE CAPABLE OF LEARNING ALGEBRAIC STRUCTURES, HINGES ON ONE'S JUDGMENT OF THE VALIDITY OF THE INDIVIDUAL TEST USED AS THE MAJOR CRITERION. FURTHER STUDY AND LONG-RANGE EXPERIMENTATION WERE RECOMMENDED. (JC)

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*A project carried out in contract with
The Cooperative Research Program,
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E. R. Keislar, Principal Investigator
Robert C. Crawford, Project Consultant*

University of California, Los Angeles, December, 1961

U. S. DEPARTMENT OF HEALTH, EDUCATION AND WELFARE
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FOREWORD

This year-long project in teaching modern mathematics to first graders through teaching machines represented a converging of the interests of the principal investigator and the full-time project consultant, Mr. Robert C. Crawford. Mr. Crawford, for years an outstanding mathematics teacher at Santa Monica High School and currently at Santa Monica City College, has always stressed the importance of structural and creative thinking. When his attention was drawn to the importance of early preparation in mathematics, he became interested in developing a curriculum in modern mathematics at the first-grade level.

The principal investigator had for some time been conducting research on the use of programmed instruction. He had prepared programs and conducted studies in the teaching of understanding in arithmetic and, in co-operation with Professor John D. McNeil, the teaching of fundamental concepts in science with young children. Such investigations suggested the importance of additional research to explore the possibilities in teaching basic principles at an early age.

In co-operation with Hughes Aircraft Company, an instructional methodology had been developed for programming instruction with young children. This methodology, employed in the present study, had been demonstrated as a useful research tool since it provided for replicability of instructional treatment and recording of pupil performance. *TTT (ing also)*

As a full-time consultant, Mr. Crawford was given the responsibility for directing the present investigation. He must be given sole credit for the preparation of the semester-long program in algebraic structures for first grade children. He made arrangements for and supervised the conduct of the experiment in the schools, a task involving careful attention to myriads of details.

Appreciation must be extended to a number of individuals who contributed measurably to this project. Miss Natalie Wilson, who assisted Mr. Crawford in the preparation of the program, rendered conscientious and invaluable service throughout the year. Several research assistants were employed on a half-time basis.

The experiment could never have been conducted without their patience, sensitivity, and understanding of the importance of experimental control. These included Mrs. Carol Fry, Mr. Kenneth Goodman, Mr. Gus Cheney, Mrs. Gloria van Deusen, and Mr. Percy Krich. Mrs. Barbara Robertson was invaluable in her competent assistance with respect to secretarial and administrative matters. Among the others who participated in the project as interviewers and assistants were Anne Simon and Molly Gorelick. In the analysis of data by the principal investigator and his preparation of the final report, the following individuals have rendered invaluable service: Mrs. Carolyn Stern, Mrs. Yasuko Filby, and Miss Lynne Bishoff.

To the schools of Santa Monica, who so enthusiastically co-operated in this research project, special thanks are extended. Mr. Glen Goodwill, Superintendent, Mr. Fred Zannon, Assistant Superintendent, and Julius H. Stier, Director of Research, have taken a keen interest in co-operating with the University in this project. A special word of appreciation must be given to Mrs. Sylvia Jordan, the Principal of the Franklin School, and Mrs. Nelda O. Sledge, the Principal of the Roosevelt School. They willingly reorganized schedules, arranged for space, and handled a variety of problems which a research study of this type invariably entails. To the teachers also must go a hearty expression of thanks. Mary Lou Drake, Thelmagene Simpson, Marie Steward, and Judith Traller, of the Franklin School, and Avalon Atteridge, Dorothy Hackett, and Betty Walters, of the Roosevelt School, felt the impact of the requirements of the experimental design most keenly. They willingly co-operated in every way with the experimenters who required that a certain number of children be sent from class every twenty minutes. Their ability to take this added confusion in stride is testimony to the skill, patience, and objectivity which are marks of good teachers. Mr. Vincent Carrell, the Principal of the John Muir School, and Mr. Paul van Alstine, Principal of the John Webster School, also took a keen interest in the project.

Appreciation must be extended to Hughes Aircraft Company who loaned to the project eight units of the Videosonic Teaching Machine, Experimental Model. This device was a prototype and has now been completely replaced by more advanced equipment through a long period of careful research and development.

This project was supported by a contract with the Co-operative Research Branch of the U. S. Office of Education, DHEW SAE 8998 Number 1090, August 15, 1960 - February 15, 1962.

This study explored the possibility of teaching mathematics in terms of concepts and principles previously reserved for considerably more advanced systems to children in the first grade. The primary and immediate objective was to teach the child to view the world of numbers in terms of patterns of relationships. The major assumption here is that if children can learn the basic concepts in a flash, further study of the area would be facilitated. If evidence can be obtained that children can indeed be successfully taught in this fashion, the current curriculum revisions will be given important direction.

The introduction of algebraic structures at the first grade level represents an extension of the movement to teach fundamental concepts and principles at earlier stages of preparation as a discipline. Algebraic structures are now being taught at the junior high level by the School Mathematics Study Group, but different approaches have already begun to dominate the mathematics curriculum in the elementary school.

Up to now, a large number of teachers and classes were required to participate what actually might happen if proposed curricular alterations were adopted. By using the technique of programmed instruction with its high degree of control and

CHAPTER ONE

INTRODUCTION

The state of healthy ferment in which education finds itself today is a product of many forces. One of these is the national and international pressure for greater intellectual competence. A second is the demand for a more efficient reorganization of the curriculum stemming from a rapid expansion of knowledge in all disciplines. The interest shown by subject matter experts in the new sequencing of topics is reflected at all levels of education. Sciences dealing with the learner are making important contributions. Finally, the introduction of technology into the school has made drastic changes in the traditional classroom setting. The present investigation has been influenced by several of these movements.

This study explored the possibility of teaching mathematics in terms of concepts and principles ordinarily reserved for considerably more advanced students, to children in the first grade. The primary and immediate objective was to teach the child to view the world of numbers in terms of patterns of relationships. The major assumption here is that if children can learn the basic concepts in a field, further study of the area would be facilitated. If evidence can be obtained that children can indeed be successfully taught in this fashion, the current curriculum revisions will be given important direction.

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Up to now, a large number of teachers and classes were required to anticipate what actually might happen if proposed curricular alterations were adopted. By using the technique of programmed instruction with its high degree of control and

replicability, large experimental populations and extensive special training of teachers become unnecessary.

The type of program used in this investigation requires that the subject matter to be taught be carefully analyzed into Cartesian-like steps. This requires an intensive analysis of the subject, as well as a clear-cut specification of each problem posed to the child. We are thus provided with an operational definition of the instructional procedure. The program was presented through the medium of a teaching machine so that each child received exactly the same treatment. In one of the two schools used for this experiment, subjects could move at their own pace without individual tutors. The teaching was carried on in a standardized fashion and in a relatively economical manner. The situations presented to each pupil were carefully recorded. A record was also kept of the child's responses to each problem, permitting further analyses and revisions of the treatment, i.e., the instructional program.

There was no implication in this study that the instructional procedures might be adopted in toto by a school since the cost of such technological methods would be fairly high in comparison to the cost of usual group instruction.

Criteria for Selection of Content

The problem posed by this investigation was whether first grade children are capable of learning mathematics in terms of algebraic structures. It should be recognized that simply because young children are able to master a topic is not in itself sufficient reason to introduce it at this age level. Dr. John D. McNeil (Keislar and McNeil, 1961) has suggested that the appropriate content for any grade level should be determined only as a result of asking a variety of questions.

These include:

- "1. Is the proposed objective something for which agencies other than the school can or should be responsible?

- "2. Is it something which the child acquires without systematic and deliberate instruction anyway?
- "3. Will it facilitate both directly and indirectly learning in later grades?
- "4. Will it enable the learner to use his experiences outside of school in a more educative fashion?
- "5. Is it economical to teach in terms of the resources of the school including personnel and their abilities?
- "6. Will it increase his ability to generate new hypotheses, to conceptualize alternatives and their consequences?
- "7. Can it be taught to the child efficiently, considering the child's abilities and interests?"

This investigation attempted to find an answer to the last question only. The other questions should also be investigated before any drastic changes in the curriculum are instituted.

Experimental Curriculum Programs in Modern Mathematics

In recent years a number of challenging curriculum developments have taken place. Foremost among these is the School Mathematics Study Group, under the direction of Professor E. G. Begle (1958), with headquarters at Yale University. Originally concerned with the curriculum in mathematics for the high school level, they have extended their field of interest into the junior high school and intermediate grades and are currently in the process of developing a curriculum at the primary level. The ultimate goal is to develop an integrated sequence of mathematics learnings from the kindergarten through the twelfth grade.

The primary emphasis in this revision has been placed upon teaching students the structure of mathematics. The approach includes the introduction of number theory, sets, metric and non-metric relations in geometry, the real number system

as a progressing development, elementary statistics, and many other topics ordinarily introduced at a much later level.

An important aspect of this program is the development of texts and other curricular materials. An extensive project in the creation of programmed textbooks is now under way. Also, the training of teachers in the field of modern mathematics has been of particular concern to the SMSG.

There are many other similar programs. These include the University of Illinois Committee on School Mathematics, the University of Maryland Mathematics Project, Greater Cleveland Mathematics Program at Bowling Green State University, Secondary School Curriculum Committee of the National Committee of Teachers of Mathematics, Commission on Mathematics of the College Entrance Examination Board, the Madison Project at Syracuse University, Ball State College Mathematics Program, and the Texas Commission to Study Mathematics.

In general, these diverse programs are characterized by certain common approaches. The emphasis is upon helping the child to discover for himself the fundamental principles of mathematics. The curriculum is sequenced so as to provide the necessary familiarity with the subject matter area, which insures that the learner will be successful in his endeavors without excessive trial and error. Mathematics is taught as a field where the major satisfactions stem from this succession of insights. Social utility, while not neglected, is given secondary emphasis. The learner is helped to view the problems in mathematics much as a game or puzzle whose solution requires an understanding of the basic pattern. In such a process, skills and mathematical facts are introduced within a logical framework.

Although most of these programs begin at the secondary level, there has been a growing recognition of the importance of introducing curriculum revision from the very beginning of the child's schooling. The present investigation was an attempt to explore the possibility of introducing algebraic structures as early as the first grade level.

Teaching Basic Principles to Young Children

Considerable interest has been expressed in the possibility of teaching fairly abstract concepts and principles at the younger levels. For example, Lewis Eigen (1959) at the Collegiate School has prepared an instructional program to teach set theory to children in the intermediate grades. Keislar and McNeil (1961) explored the possibility of teaching primary grade children certain elements of molecular theory. Seeger (1960) has prepared a curriculum for teaching physics in the first grade. Suppes and Hill (1961) at Stanford University are teaching bright children the subject of logic in the intermediate grades. Suppes (1961) has also prepared a course in arithmetic in terms of set, and a program in geometry for children in the primary grades has been developed by Suppes and Hawley (1961).

Bruner and Dienes (Annual Report, 1961) at the Center for Cognitive Studies at Harvard University, have been exploring whether young children can be taught the associative, distributive, and commutative principles of mathematics, as well as a number of other mathematical concepts. The method used was largely observational, but highly encouraging results were obtained. They were able to "teach nine-year olds to distinguish between finite games with and without mathematical group structure, and, indeed, to construct these games."

However, these studies have not all been uniformly successful. Further experimentation and exploration should cast far more light on what can be taught children at these younger ages. The underlying assumption which characterizes most of these investigations is that basic concepts in an academic area might be introduced profitably early in the life of a school child. If a young child can grasp a principle or assumption regarded as basic to a subject, he will have at his disposal a conceptual tool with which to progress more efficiently in this field. Such abstractions should, therefore, be acquired by the learner as soon as possible. From the point of view of curriculum construction, such content should not be postponed simply because of the presumed difficulty of providing appropriate learning opportunities. It may be advisable to introduce such topics before the

child has acquired interfering notions or habits of thinking which may later have to be unlearned.

Bruner and Dienes have concluded "that any subject can be taught to any child in some honest form" (Bruner, 1960, p. 52). In his report on the Woods Hole Conference, Bruner (1960) has given the following summary:

"... the curriculum of a subject should be determined by the most fundamental understanding that can be achieved of the underlying principles that give structure to that subject. Teaching specific topics or skills without making clear their context in the broader fundamental structure of a field of knowledge is uneconomical in several deep senses. In the first place, such teaching makes it exceedingly difficult for the student to generalize from what he has learned to what he will encounter later. In the second place, learning that has fallen short of a grasp of general principles has little reward in terms of intellectual excitement. The best way to create interest in a subject is to render it worth knowing, which means to make the knowledge gained usable in one's thinking beyond the situation in which the learning has occurred. Third, knowledge one has acquired without sufficient structure to tie it together is knowledge that is likely to be forgotten. An unconnected set of facts has a pitifully short half-life in memory. Organizing facts in terms of principles and ideas from which they may be inferred is the only known way of reducing the quick rate of loss of human memory.

"Designing curricula in a way that reflects the basic structure of a field of knowledge requires the most fundamental understanding of that field. It is a task that cannot be carried out without the active participation of the ablest scholars and scientists. The experience of the past several years has shown that such scholars and scientists, working in conjunction with experienced teachers and students of child development, can prepare curricula of the sort we have been considering."

In teaching arithmetic in terms of fundamental relationships, the Cuisenaire method has been extensively followed in Europe and more recently in Canada. This approach is becoming more widely used in schools in the United States. The details of the method, which involves the use of rods of different colors to represent different lengths, is more fully described in the next chapter.

Cuisenaire and Gattegno (1957) claim that their technique "is the first

rational approach to teaching arithmetic." (p. vi) According to this point of view, the comprehension of "primitive mathematical relations" is facilitated through experience with the manipulation of the rods, by means of which children discover important properties of numbers. They maintain that the use of such "semi-abstract" materials offers a highly flexible and creative method for the development of mathematical insights.

Although Gattegno (1959) has outlined a suggested sequence of problems using these rods, teachers are left free to present the materials in other ways they feel are also appropriate. Most of the work with the Cuisenaire rods has been of a demonstration nature, but some experimental studies, such as that of Howard (1957), have shown the value of this approach.

Catherine Stern (1949) has developed an approach to the teaching of arithmetic in which the child discovers "the relations between the numbers of our number system." This Structural Arithmetic involves the use of number bars which provide opportunities for experiments on the part of the child. The number bars of various lengths are marked off so as to measure one unit, two units, and so on.

Development of Logical Thought in the Young Child

Much of the research activity carried on by individuals studying the intellectual processes of children has stemmed more or less directly from the work of Jean Piaget. In his insightful study of the development of logical processes in children, Piaget (1953b) speaks of the psychological development of operations which he describes as "actions"¹ which are internalizable, reversible, and co-ordinated into

¹ "They are actions, since they are carried out on objects before being performed on symbols. They are internalizable, since they can also be carried out in thought without losing their original character of actions. They are reversible as against simple actions which are irreversible. In this way, the operation of combining can be inverted immediately into the operation of dissociating, whereas the act of writing from left to right cannot be inverted to one of writing right to left without a new habit being acquired differing from the first. Finally, since operations do not exist in isolation they are connected in the form of structured wholes. Thus, the construction of a class implies a classificatory system and the construction of an asymmetrical transitive relation, a system of serial relations, etc. The construction of the number system similarly presupposes an understanding of the numerical succession: $\underline{n} + \underline{1}$." (1953b, p. 8)

systems characterized by laws which apply to the system as a whole."

Piaget maintains that there are four main stages in the psychological development of operations. The first period, the sensory-motor period, extends from birth to two years of age. The beginnings of an operation of reversibility and associativity are to be noted in the motor behavior of the child in space.

Piaget calls from two to seven years of age the period of pre-operational thought. With the advent of language, actions can become internalized into thoughts. The child is not yet able to deal with reversible operations and does not display any concept of conservation. For example, when a number of beads are poured from one glass into another of different size and shape, he believes that the quantity of beads is changed in the process. Two sticks of equal length are equal if their end points coincide, but they are not equal if one is moved slightly to one side or the other. The child's judgment of quantity lacks transitivity. He may note that two quantities, A and B, are equal, and similarly that B and C are equal, but is unable to conclude that A is equal to C. Piaget calls this period "pre-operational" because he finds that the use of operations, as defined earlier, appears first between seven and eight years in a concrete form.

The period of concrete operations extends from the age of seven to eleven. The child is now able to deal with objects in a manner which shows the characteristic of reversibility. The first operational system is that of classification, in which the child can subsume one class under another. The second equally important system is that of seriation, i.e., arranging elements in a series according to a specific attribute such as length. Now systems having multiplicative character begin to develop. The child can classify the same object according to two attributes, e.g., shape and color. The thought of the child is restricted, however, in that the operations are still "tied" to the concrete objects to which they are related. These concrete operations are not yet combined by the child into a "single, structured whole." The child, therefore, is not yet engaging in what can be called a formal logic.

The final period is one of propositional or formal operations. It begins about the age of eleven or twelve, stabilizes about fourteen or fifteen, and then develops into an adult logic. Not until this final stage, beginning at the age of eleven or twelve, is the child able to reason by hypothesis and to draw logical conclusions from purely hypothetical data. The child is no longer anchored to concrete objects. He now begins to theorize and establish or verify postulated relationships between things, events, and ideas. Instead of just committing pieces of information to memory, he will be able to draw implications from various statements and synthesize these implications. He will be able to perform abstract, propositional operations of implication, disjunction, incompatibility, etc.

During this period, the child deals with a group of operations called "operational schemata." One of these involves combinatorial operations in general, such as combinations, permutations, and aggregations. A second operational schemata deals with proportions. Piaget points out that this fourth period

"...includes two important acquisitions. Firstly, the logic of propositions, which is both a formal structure holding independently of content and a general structure co-ordinating the various logical operations into a single system. Secondly, a series of operational schemata which have no apparent connection with each other nor with the logic of propositions." (1953b, p. 22)

The first grade child is viewed, therefore, as pre-logical, or in the beginning stages of transition to the next stage. But Piaget finds many possibilities for advanced thinking in embryonic form. For example, he has shown a parallel between the young child's concepts of objects, causality, velocity, and space, and the thinking of advanced theoretical physicists (1957).

Piaget (1953a) has distinguished three types of comparisons of perceived quantity, made without actual measurement. Using a variety of materials, including liquids, sticks, and beads, he found three age-related, hierarchically-ordered stages. Children at the first stage, usually at four years of age, succeeded in comparisons involving gross relationships, such as "larger than," "longer than," etc. At the second stage, usually age five, children succeeded with intensive comparisons, perceiving quantity relations taken two at a time, e.g., "longer and wider," "taller

and thicker," etc. It wasn't until the third stage, age six, that children were able to make simultaneous comparisons of intensive plus extensive quantity relations between objects, such as "x is half of y," "x is twice y," etc.

Although Piaget himself has not undertaken rigorous experimentation, many other investigators have found his work to be a fruitful source of hypotheses. For example, Elkind (1961) claims to have validated experimentally Piaget's hypotheses regarding the development of quantitative thinking. He has found that age groups differed significantly in their success in comparing quantity: the type of quantity and the type of relationship significantly affects successful comparison.

Professor Inhelder at the Harvard Center for Cognitive Studies (Annual Report, 1961) is currently studying techniques for speeding the transition of children from one cognitive level to the next.

The Formation of Concepts in Young Children

The development of concepts plays a central role in all teaching. Kendler (1961) defines concept formation very simply as "the acquisition or utilization, or both, of a common response to dissimilar stimuli." As he indicates, simple stimulus discrimination may be used to deal with this process. For example, a subject learns to respond to a single property or combination of properties of a stimulus. The common feature of a class of stimuli which gains control over a particular response is referred to as an abstraction, a concept, or a universal. Other investigators have viewed concept formation as essentially the result of mediated discrimination. The various stimuli belonging to some class acquired stimulus equivalence by virtue of a common mediator. There is a common learned response which permits the learner to treat all members of a particular class alike. The nature of this mediating response may be either perceptual or verbal.

The teacher must decide whether to offer a large variety of illustrations of a concept, or to work far more intensively with a few. While Hull (1920) found that some acquaintance with a large number of concepts is more effective than twice

as much familiarity with half the cases, Levine and Harlow (1959) found that in the acquisition of learning sets it made no difference whether the individuals trained thoroughly on a few problems or were exposed only once to many. The over-all number of exposures to the instances was what seemed to count.

Morrisett and Hovland (1959) suggested that transfer in learning sets involved learning within the problem and generalization between the problems. They found that permitting both processes to operate, i.e. a relatively high degree of learning on a moderate number of problems seemed to be the most effective procedure.

Research by Bourne (1957) and by Bruner, Wallach, and Gallanter (1959) supports the position that concept formation is more difficult when the instances of the concept include irrelevant properties. It would suggest that in teaching children concepts it is wise to facilitate responding to the relevant property by using very simple instances. The addition of irrelevant, variables especially if familiar to the child may be inappropriate at the early stages of learning.

In dealing with concept formation, Dienes (1961) prefers to speak of processes of abstraction and generalization. An abstraction is formed if "situations of the same structure which at first appear different are classed together in one class." In this process the child draws out from a number of different situations that which is a common feature.

Generalization proceeds from classes to classes, while abstraction proceeds from elements to classes. Primitive generalization is the "extension of the class to which a certain attribute applies." A mathematical generalization occurs when some principle or property of Class A is extended to a broader Class B, provided that Class B includes an isomorphic image of the relevant features of Class A. Thus, a child may learn by using small numbers that the order in which numbers are added has no effect upon the sum. He has broadened the class to which this principle applies - an example of primitive generalization. But, when the child applies a principle learned in relation to the class of numbers in arithmetic only, and then extends this principle to classes of both positive and negative elements, he is

engaging in mathematical generalization.

Role of Verbalization

A central issue in the field of developing concepts and generalizations is whether the teacher should ask the child to state the rule. Hendrix (1947), for example, reports a study in which three groups of college students were taught by different methods. In one group, the teacher stated the principle with a clarifying illustration and then had the students apply the principle to several examples. The members of the second group were presented with problems where the principle applied but without any statement of the principle. The third group, when it appeared that they had acquired the concept were asked to state the rule they had discovered. They were required to modify their statements until the principle was accurately verbalized. Two weeks later, all subjects were given an opportunity to use the principle in the solution of items scattered throughout a test. The highest transfer effects were achieved by the group who were not required to verbalize their solutions. The group who were told the rule did the poorest. Gertrude Hendrix concludes that un verbalized awareness provides superior transfer than either teaching the rule directly or formulating it in words. More recently (1961) she has elaborated on this conclusion.

There is ample evidence that concepts may be learned even though the subject is unable to verbalize the basis for his responding in a similar fashion to a diversity of stimuli. Hull (1920) in his classic experiment on concept formation clearly found this to be true; and in a more recent study by Giddan and Eriksen (1959) subjects generalized an acquired response bias to a concept formation task even though they were unaware of this newly acquired behavior.

On the other hand, laboratory research has equally clearly indicated the value of verbal responses as mediators. Goss and Moylan (1958), for example, have shown that in learning concepts prior experience in labeling the instances is valuable. Norcross (1958) has shown that the distinctiveness of the labels is important. A Russian investigator, Liublinskaya, (1957), found that the learning of concepts is greatly facilitated by the use of verbal mediators.

We should note that apparent discrepancies such as these may stem in large measure from the particular language used. In dealing with young children, words expressing the rule or the basis for classification may not function for the child simply because these words are not adequately associated with the objects, attributes, or operations. Mathematical language such as " $A + B = B + A$ " may be taught to young children so as to become an appropriate mediator, whereas, expressing this principle in everyday words may be quite ineffective.

Dienes and Bruner (Annual Report, 1961), as a result of their investigation of the abilities of young children to learn mathematical principles, have suggested the following:

1. The Construction-Analysis Theorem implies that having the child construct a concrete "embodiment or two" to represent an idea is a better instructional procedure than requiring the student to decompose or analyze an "embodiment presented to him ready-made."
2. The Notation Theorem suggests that if a learner is to grasp a "sequence of construction steps" the results must be presented in a fashion which permits the learner to observe all of them simultaneously as illustrated by a graph.
3. The Operational Theorem maintains that in transforming successive steps into a simultaneous notation, "a special cognitive act" is required; this does not take place at random or with a passive learner.
4. The Game Theorem asserts that symbolic activities like mathematics are most effectively learned when extrinsic rewards and punishments are reduced to a minimum.

Discovery

Discovery has been widely advocated as a method for promoting in the child an understanding of concepts and principles. Hendrix (1959) has pointed out that having a child discover an abstraction for himself is far superior to having him told either the rule or the basis of classification. Bruner (1961) has presented some evidence that permitting children to select their own mediators as a basis for remembering paired-associate material was superior to supplying a matched group with the mediators selected by the first group.

On the other hand, Ausubel (1961) maintains that there is little research support for the technique of "learning by discovery." He points out that in Bruner's experiment the children in the "discovery" group had selected mediators with strong associative values. When these mediators were supplied to children with different prior experiences, they did not carry the same strength of association. The difference in results obtained, therefore, is simply a function of differences in associative strength of the mediators used. Ausubel concludes that the most effective formal learning takes place when the material is so organized that the child acquires appropriate mediators of strong associative value.

The advantage of the discovery method is that the child selects through a type of mental trial and error those associations which are most relevant, and at the same time the strongest. In the same sense he is engaging in a process of testing hypotheses which he formulates through his own mediating behavior. These hypotheses need not be verbal in nature; they might be perceptual - that is, "ways of looking at the situation." But as the child's experience leads him to the more formal parts of the subject, the use of verbal mediators becomes more and more important. In any event, the child must demonstrate that he has learned the concept or principle by successfully meeting new problems of the same kind or ones involving the same principle.

It may be argued that if we knew ahead of time exactly what the strongest associations were for a child, these might well be evoked directly and conditioned

to the appropriate class of situations. By removing this prompt, and requiring the child to continue to solve problems of this type, he may have acquired exactly the same understanding as the child who was permitted to select his own associations. It is even possible that this kind of "guided discovery" might provide the child with better mediators than he could otherwise supply for himself. As Ausubel points out (1961), most high school students would sit in a bath for years and still not arrive at Archimedes' principle!

But the proponents of discovery have never suggested that the problems posed for the child be too difficult. The sequence of problems should be arranged so that the child can learn at one stage concepts and principles which he will need to know in order to discover new ones at a later stage. When this is analyzed more closely we see that it involves very much the same types of principles advocated by many persons in the field of programmed instruction. Briggs (1960) has suggested the use of indirect cues (rather than direct prompts) to bring about the appropriate mediating responses which would then lead to the desired overt behavior.

The process of discovery, therefore, may be effectively utilized in a programmed learning situation. The child, through the sequence of frames, is helped to discover the patterns for himself and may be expected to test them so as to arrive at the appropriate concept or principle. If the continuous record kept of the child's performance (one of the most important advantages of programmed instruction) indicates that the sequencing is too difficult in the light of pupil performance, subsequent improvement of the program would be indicated.

Intrinsic Reinforcement

In the field of programmed instruction, considerable interest has been expressed in the possibility of preparing frames in a program where the response itself supplies reinforcement. Such intrinsic reinforcement may result from the manipulation of environmental variables to produce a reinforcement stimulus (as in finishing a picture puzzle or building a tower with blocks). As Glaser (1962) has pointed out "In many well-constructed frames there is little doubt that the student,

after having made a response, can check or monitor himself regarding the correctness of his response without the necessity for frequent response confirmation frames." Glaser points out that this concept of intrinsic reinforcement is similar to the term "autotelic" coined by O.K. Moore. Autotelic frames must be of the appropriate difficulty, neither too hard nor too easy. Homme¹ has indicated current work in the use of humor as an intrinsic reinforcer. He has been exploring the possibility of using low probability responses for this purpose.

An important feature of the "discovery" method may be precisely that intrinsic reinforcement is supplied and hence, motivation is heightened. For example, Kersh (1958) found that although a no-help group performed significantly better on the post test than a "directed reference" group, this superior performance could be explained in terms of motivation.

The Instructional Program in Relation to Prior Work

Since the major hypothesis of this investigation was that first-grade children have the ability to learn mathematics in terms of algebraic structures, an auto-instructional program was prepared by Mr. Robert Crawford to provide the appropriate learning opportunities. (The program is described in the next chapter.)

In terms of the review of the literature, the following characteristics of the program should be noted.

1. Concepts and principles were presented in their simplest possible form. Thus, all illustrations were in terms of length and color through the use of the Cuisenaire rods. Examples from the everyday life of the child were not used since these would introduce irrelevant stimulus dimensions which might be confusing.
2. The problems were presented in a sequence which was designed to facilitate independent discovery of the concepts and principles being taught. The learner

¹ From an unpublished address "Mass Testing of Linear Programs" given on January 15, 1962, at the Third Western Conference on Programmed Learning, Systems Development Corporation, Santa Monica, California.

was never told this concept or principle; he was expected to formulate it or perceive it for himself. Notation and terminology, being arbitrary conventions, were presented in straightforward fashion.

3. Insofar as possible, the program was constructed on the principle of intrinsic reinforcement. It was true that the child was informed by the program whether his answer to each problem was correct; in fact, he had to get each problem right before he could proceed to the next. But the child could frequently tell by his own manipulatory activity whether his answer was correct before registering a response: the right block "fitted" and the wrong one did not. Furthermore, it was assumed that finding a general solution to a class of problems, i.e., forming an appropriate mediator, would constitute an automatic reinforcer: it would give the child a "feeling of satisfaction" which would be intrinsically rewarding. Every effort was made to present the field of mathematics as a "game" or "puzzle" rather than as a task to be fulfilled to please the teacher.

4. Instead of presenting complex material and then helping the learner analyze it, the program sequence first provided mastery of simple elements with which the more advanced patterns could later be constructed.

5. Verbalization of the postulates was not required as a solution of problems in the program (nor in the evaluative criteria used in the post-tests). Each child was left to develop his own mediating process through unverbalizable perceptions or through the addition of verbal mediators.

CHAPTER TWO

A FIRST GRADE PROGRAM IN ALGEBRAIC STRUCTURES¹

Traditional mathematics has been created in the main by individuals who were attempting to satisfy certain practical requirements and, thus, with few exceptions, has lagged the culture. Most of these practical requirements grew out of a desire to structure reality as observed through sense perceptions and the extension of sense perceptions. This traditional mathematics, even though very useful, usually lacks rigor and a sound logical basis. Thus it is to be expected that as mathematics increases in maturity, attempts will be made to reorganize this utilitarian mathematics in the furtherance of greater rigor and sounder logical bases.

Today, modern mathematicians, as they attempt to set up an absolute logic or super-theory, tend to frown upon intuition as a basis of mathematics. These modern mathematicians are divided into two philosophical camps: the logicalists and the formalists. The logicalists, exemplified by Bertrand Russell, contend that mathematics is a branch of logic. The formalists, exemplified by David Hilbert, contend that mathematics is a collection of formal, symbolic systems. In spite of the pressure exerted by these two camps, there still remains a group of intuitionists who would base all mathematics on the natural number, i.e., the counting numbers.

Each point of view has merit and there is a time and a place when one position may be more satisfactory than the other two. There might also be times when a combination of these philosophies would provide a better answer. An individual's original experiences with mathematics must be of necessity appeal to intuition. However, if these first experiences are carefully planned by a qualified instructor, they can very quickly lead to the formation of a logical abstract structure which can be used as the basis of understanding experiences that come later. Eventually it is to be

¹

This chapter was written by Robert Crawford to indicate the philosophy and some of the principles adopted in the preparation of the program.

hoped that the individual will see the possibility of creating other abstract structures without regard to reality or experience.

Psychologically, each individual has within himself a strong desire to structure - an urge for order. If at any stage in his intellectual maturing he is confronted by a situation that is almost, but not completely, structured and if, further, his own previous structure has been of the form that can be easily extended, it is probable that he will tend to extend his previous structure to include the present situation. The earliest type of structuring comes as a result of actions and perceptions involving concrete objects. (Abstract verbalizations come much later.) Therefore, it is highly desirable from a learning point of view that the learner be first exposed to situations involving concrete objects that he can manipulate and, thus, discover relationships between the elements. One must, however, be very wary lest random actions and perceptions be mistaken for the desired learning. The desired actions and perceptions must be carefully planned and programmed so that there is a high probability that the learner will structure any given situation in such a way that it might serve as a foundation for future structuring situations. The more "naked" the variables² with which the learner works, the better the chance of communication between the situation-planner and the learner. Many meaningful situations (that is, meaningful to the instructor) present so many superfluous variables and embellishments to the student that they distract his attention from the intended focal point of the experience.

A set of elements that differ only in length and color, for example, are ideal for first experiences. One such available set of elements is the Cuisenaire rods. (See Figure 1.) In this set, rods of the same color are the same length and rods of the same length are the same color. The variables of color and length are "naked," in that they are free from distracting embellishments. There can be no ambiguity when one refers to either the variable of color or the variable of length. The

²I.e., variables purely denotative, not connotative.

visual and tactile senses serve to communicate information about that and only that variable under consideration.

Many studies indicate that young children have an intense interest in the world of color. This, no doubt, accounts for the early classification of colors. With this knowledge of color as a starting point, additional relationships can be established by using color as a cue.

Since the elements of the set are concrete, the abstract is put in terms of reality by example. By actions and perceptions with these concrete elements, thoughts that would otherwise be too abstract for the young learner can be communicated more directly. This concreteness also makes it much easier to pinpoint EQUIVALENCE relationships between elements or combinations of elements.

First grade pupils have the capacity to understand abstract relationships if they are presented in concrete form. Abstract symbols can be used to communicate operations if they are associated with experience in the manipulating of concrete objects. It is not the lack of capacity on the part of first graders, but rather, the lack of ability on the part of the educator to communicate that has retarded the young learner in dealing with abstractions.

These elements can serve for the building of a sound logical foundation which lends itself to deductive extension. This structure not only explains the present situation, but is sound mathematically and logically for explaining future situations.

Similar sets of elements, concrete and "naked," can be invented that serve as sense stimulators to help the young learner build a sound logical structure. For example, a set of elements that differ only in weight and color hold great potential.

If we define the binary operation of adding two elements as the placing of two elements end to end, the pupil will be introduced to and gain a feeling of UNIQUENESS and CLOSURE in that for the selected problems in his first experiences there will always exist one and only one element of the set with which he is working that will have the same length as the two elements placed end to end. However, a deeper sense of closure will be experienced when the student for the first time discovers that no

element in his set is equivalent to the sum of the two given elements. It is at this time that he will experience a need for extending his present set of elements in order to satisfy closure. Later similar situations which will require further extensions will make it possible for the learner to expand his basic mathematical structure to include larger and larger fields.

We now have a set of elements which is closed under a single-valued binary operation (addition), and we hope to have the learner verify the following postulates through actions and perceptions using these elements:

1. Associative:

If three or more elements are to be added, the sum is the same regardless of whether you add the sum of the first two elements to the third element OR whether you add the first element to the sum of the last two.

2. Commutative:

For any two elements a and b , the sum $a + b$ is the same as $b + a$.

3. Zero:

There exists in the set an element 0 such that if added with any other element, the sum is the other element itself, i.e., $a + 0 = a$.

4. Additive Inverse:

For every element in the set there exists an element in the set such that the sum of these two elements is 0 .

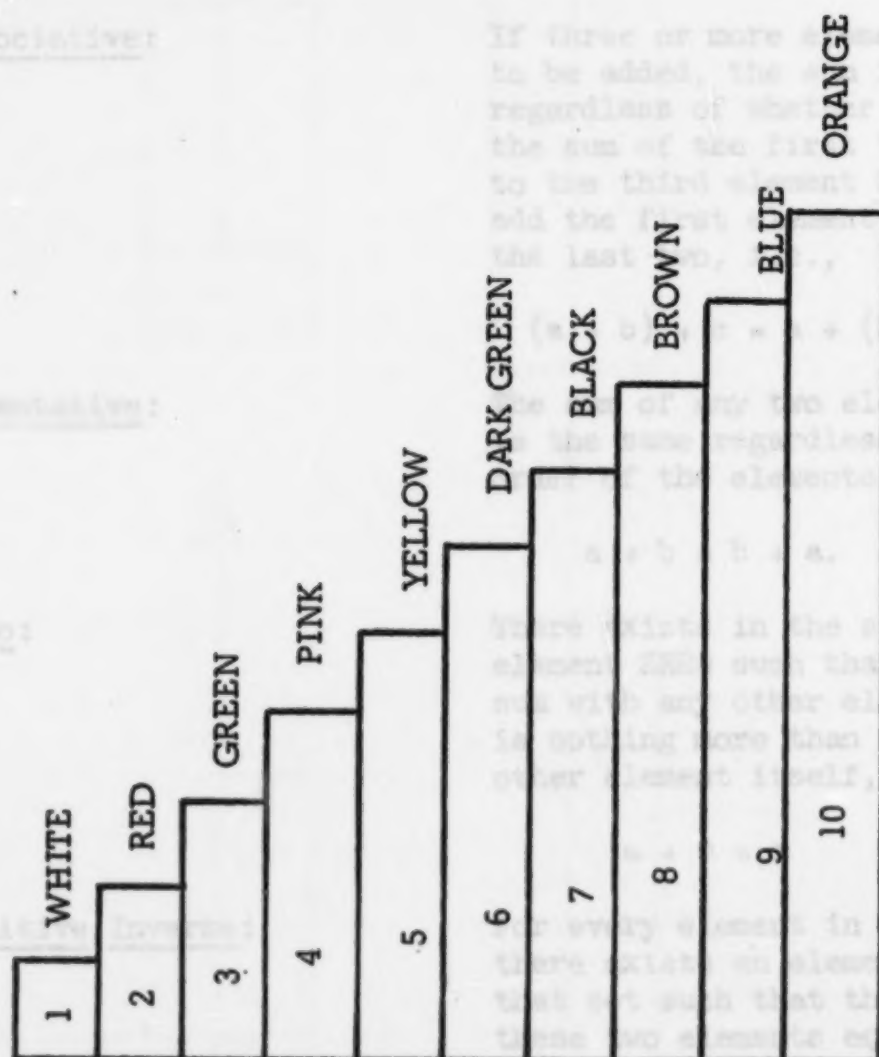


FIGURE 1.

A Schematic Representation of the Cuisenaire Blocks

As our second binary operation on this set we shall define multiplication. As greater clarity may be achieved here diagrammatically rather than verbally, multiplication is defined by the typical diagram in Figure 2. The binary operation of multiplication on the elements A and B as shown in the diagram is accomplished by first collecting " A " B -elements, and then placing these B -elements end to end to determine a third element C which is the same length as these B -elements placed end to end.

element in his set is equivalent to the sum of the two given elements. It is at this time that he will experience a need for extending his present set of elements in order to satisfy closure. Later similar situations which will require further extensions will make it possible for the learner to expand his basic mathematical structure to include larger and larger fields.

We now have a set of elements which is closed under a single-valued binary operation (addition), and we hope to have the learner verify the following postulates through actions and perceptions using these elements:

1. Associative:

If three or more elements are to be added, the sum is the same regardless of whether you add the sum of the first two elements to the third element OR whether you add the first element to the sum of the last two, i.e.,

$$(a + b) + c = a + (b + c).$$

2. Commutative:

The sum of any two elements is the same regardless of the order of the elements, i.e.,

$$a + b = b + a.$$

3. Zero:

There exists in the set an element ZERO such that its sum with any other element is nothing more than the other element itself, i.e.,

$$a + 0 = a$$

4. Additive Inverse:

For every element in the set there exists an element in that set such that the sum of these two elements equals zero, i.e.,

$$a + x = 0$$

As our second binary operation on this set we shall define multiplication. As greater clarity may be achieved here diagrammatically rather than verbally, multiplication is defined by the typical diagram in Figure 2. The binary operation of multiplication on the elements A and B as shown in the diagram is accomplished by first collecting "A" B-elements, and then placing these B-elements end to end to determine a third element C which is the same length as these B-elements placed end to end.

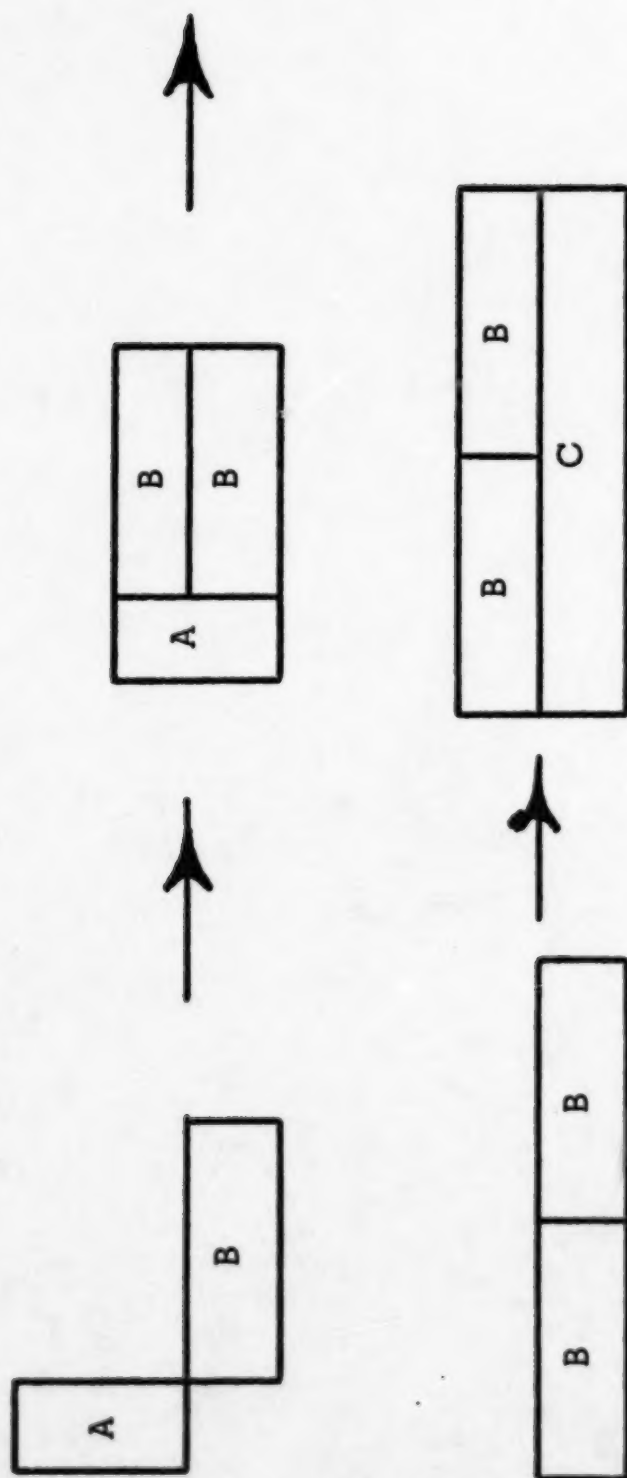


FIGURE 2.

Steps in the Manipulation of Blocks for Multiplication

Here again the pupil is introduced to and gains a feeling of UNIQUENESS and CLOSURE in that there will always exist one and only one element of the set with which he is working that will have the same length as is yielded by the two elements to be multiplied. Again by carefully planning the problems presented, the student can be placed in a situation that will require extending his present set of elements.

With this newly defined second binary operation (multiplication) the student will be asked to verify for multiplication the postulates stated above and also the following postulates:

5. Unity:

There exists in the set an element UNITY such that when this element is multiplied by any other element of the set, the product is simply the other element itself, i.e.,

$$a \cdot 1 = a.$$

6. Multiplicative Inverse:

For every element in the set there exists an element of that set such that the product of these two elements equals unity, i.e.,

$$a \cdot x = 1.$$

7. Distributive:

The product of an element by the sum of two elements is the same as the product of the single element times the first element of the sum ADDED TO the product of the single element times the second element of the sum, i.e.,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

The student is also introduced to and given a feeling of the following laws of logic through actions and perceptions with his set of elements:

1. Reflexive:

$$a = a;$$

2. Symmetric:

$$\text{If } a = b, \text{ then } b = a;$$

3. Transitive:

$$\text{If } a = b \text{ and } b = c, \text{ then } a = c;$$

where a , b , and c are elements in a given set.

In any experience many concomitant results are of equal importance with the objectives and insofar as these concomitant results are predictable they too

should be mentioned. Some such concomitant results are:

1. Law of Trichotomy:

Two elements must be ordered in only one of three possible ways:

$$a < b, a = b, \text{ and } a > b.$$

2. Indirect Reasoning:

If all possible choices contain a solution and all but one of these choices can be proven to be not-solutions, the remaining choice is a solution.

3. Set Membership:

An element either belongs to a given set or does not belong to that set.

4. Set Intersection:

The intersection of two or more sets is that set of elements that is common to the two or more sets.

5. Set Union:

The union of two or more sets is that set of elements whose members belong to any one of the given sets.

In addition to the aforementioned concepts it is hoped that the learner will read meaning into the following abstract symbols: $+$; $-$; $.$; $\div$; $=$; $>$; $<$; $()$; and X as a place-holder.

The students will be given extensive experience in equivalence relationships. They will be asked to find solution sets, if any exist, to equations and inequations. These solution sets will first be chosen from the set of sticks, then from the non-negative integers, then from the non-negative rational numbers, and eventually from the rational numbers. It is hoped that a problem presented in equation or inequation form using abstract symbols will motivate the learner to seek a solution set. This implies that the pupils will be aware of the presence of a problem when one exists and, further, that he will be able to understand that problem, and finally will desire to find a solution set.

Abstract algebraic structures will be made more meaningful to the learner through the use of geometric models. In particular, the elements of our original set will be mapped onto a number-line and this in turn will be extended to a plane with Cartesian co-ordinates. Students will, thereby, have an opportunity to

experience the difference between static elements (fixed sets of points) and dynamic elements (transformations of sets of points). Experiences will be so set up that there is high probability that the student will observe both static and dynamic patterns. In all of this, intuition will play a large part in helping the student to structure a situation that might otherwise be entirely too abstract for his stage of mathematical development.

EXTENSION is the key philosophy back of the entire program. For example, the number-line and the defined method of combining the elements in our linear vector approach should make it a very simple matter to place the student in a situation in which it is highly probable that he will discover ZERO and the NEGATIVE INTEGERS. At an even later date division can open the doors to the rational numbers. And much, much later, the number-line and the student's already developed algebraic structure will extend to include all real numbers. The number-line can also be extended very easily to two-dimensions, thereby making it possible to map onto the plane all possible real ordered number couples. This extension will then make it easier to understand equivalence sets as they pertain to handling fractions, solving equations and inequations, performing linear transformations, etc. Thus it is hoped that the student will see the relationship that exists between algebra and geometry and, we hope, appreciate the fact that a non-analytically defined geometry does not offer proof.

Mathematics as a game offers the student an opportunity to experience the THRILL OF DISCOVERY and to satisfy his desire for aesthetic perfection. Mathematics in and of itself opens doors that might never have been seen. These doors never have been seen because of the many embellishments that camouflage them. As Einstein said, "How can it be that mathematics, a product of man's mind without regard to experience, fits reality so well?". Thus, one of the main objectives of this suggested approach to mathematics lies in pointing out the fact that the present frontiers of knowledge are going to be extended primarily through pure deductive abstract reasoning as contrasted with the past and present empirical approach. The type of thinking that will

be required by all in the not too distant future will have to be vastly superior to the "I-can-explain-it-with-common-sense" type of thinking. Rigor in thinking can be sharpened by experience in forming independent consistent, exhaustive sets of postulates for a given set of well defined elements under given operations. Mathematics ideally offers this experience because the elements and operations can be defined without regard to "truth" in the real world. The "meaningfree" structures might later prove to be the vehicle or model needed to explain some as yet undiscovered or unexplained phenomena in reality.

In order to allay the fears of present teachers, school administrators, parents, and professional educators, it should be noted that the suggested curriculum contains the present traditional curriculum as a subset. At many stages of the experiment the abilities of the experimental group was compared with the abilities of the control group in the handling of traditional arithmetic skills. On the basis of the study just completed the experimental group did, in fact, surpass the control group in these traditional skills.

The program consisted of 75 daily lessons. A summary statement describing each lesson is presented below:

OUTLINE OF PROGRAM IN: ALGEBRAIC STRUCTURES FOR FIRST GRADE

- Lesson 1. Thirty-six problems in which the student is asked to match colors according to some given set of conditions, e.g. which element at the bottom of your screen is the same color as the smallest element at the top of your screen?
- Lesson 2. In this lesson, the binary operation, "end-to-end," with its unique symbol "+" called plus is introduced and some experience is given with the equivalence of notation. Also problems are illustrated in which the unique third element equivalent to the length of two elements under the binary operation of "end-to-endness" is found.
- Lesson 3. In this lesson, eighteen problems are presented in which the student is asked to find the element of his set that is the same length as two given elements placed end to end, or "added." The problems are so presented that a dynamic pattern of first commuting, then the transitive law, then the law of symmetry, again commuting, and finally the transitive law, increases the probability that a student will structure certain principles of equivalence.

- Lesson 4. Continuation of Lesson 3.
- Lesson 5. More problems as noted in Lesson 3.
- Lesson 6. In this lesson, the additive inverse is introduced and the student is asked to find which element from his set must be added to a given first element in order to be equal in length to a second given element. Here again the dynamic pattern noted in Lesson 3 is repeated.
- Lesson 7. A continuation of Lesson 6 and an extension to problems in which the problem is presented verbally only.
- Lesson 8. In this lesson the symbol "-" is introduced as is also the place-holder, "?." Problems are then presented in equivalence form using the place-holder "?" for the element that will make the "STORY" true, e.g., $5 + 4 = ?$
- Lesson 9. In this lesson, illustrations of "how to" find the element from the student's set that will make the story true when placed in place of the place-holder. Also a few problems in which the student is asked to replace the place-holder with the element that will make the story true.
- Lesson 10. More problems using "-" and "?" in stories with emphasis in this lesson on the additive inverse.
- Lesson 11. In this lesson "X" is introduced as the place-holder in place of "?" and more problems in making stories true are presented.
- Lesson 12. In this lesson, "=" is introduced as an equivalence notation for the additive inverse and problems are presented by means of this new symbol. Also an initial step is taken toward "USHERING" certain subsets of the student's set.
- Lesson 13. More problems are presented using the new "-" symbol and an evaluation of knowledge of all concepts encountered to date is made.
- Lesson 14. The evaluation of the student's learning is continued, usually by asking the student to indicate whether a story is true or false.
- Lesson 15. In this lesson, the element "ZERO" is introduced, also the initial experiences with associativity are presented. The final questions in learning evaluation are also included herein.
- Lesson 16. New names, "NUMBERS," are given to elements that have to date been identified only by color. A few problems are presented in story form in which the set from which the solution set is to be chosen is listed in "NUMBER" form only.
- Lesson 17. The remainder of the elements in the student's set are given "NUMBER" names and problems involving these recently named elements are given. Ordering of names, i.e. numbers, is mapped by means of ordering with familiar sticks.
- Lesson 18. More problems involving the new names, numbers, for the elements.

- Lesson 19. Ordering of names is reinforced and the sticks with numbers attached are now presented upon the screen for the first time.
- Lesson 20. Stories with numbered elements are presented and the student is required to find which numbered stick from the bottom of his screen he would put in place of the "X" at the top of his screen in order to make the story true.
- Lesson 21. Problems involving numbered elements under binary operations, + and -, in stories in which the student must replace the X with the element that makes the story true.
- Lesson 22. More of Lesson 21.
- Lesson 23. More of Lesson 21.
- Lesson 24. More of Lesson 21.
- Lesson 25. More of Lesson 21.
- Lesson 26. Each lesson is composed of eighteen problems involving numbered elements in which the student is asked to find the element that makes the story true. In a few cases, closure fails for the set that the student has and a NEED is created to extend the original set.
- Lesson 27. More of Lesson 26.
- Lesson 28. More of Lesson 26.
- Lesson 29. More of Lesson 26.
- Lesson 30. In this lesson, the elements are stripped of their color and only the number and the length of the element remain as variables. The pictures on the screen are in black and white. However, the students are still encouraged to discover their answers by tactile manipulation with their elements.
- Lesson 31. Eighteen problems in which the students are asked to form true stories using colorless sticks with their names, i.e. numbers, on them.
- Lesson 32. A set of problems in which ZERO is an element of the set of elements with which the student is asked to form true stories. The elements are again identified only by their length and their name, i.e. number.
- Lesson 33. For the first time in this lesson, no sticks appear on the screen and the problems are determined by the names of the elements only. Thus, the student is now working with abstract numbers in building true stories.
- Lesson 34. An evaluation of the learning on all phases to date.
- Lesson 35. Further evaluation of learning to date, including sets of elements with color, with color and name, without color but with name, with name only. Also some questions on the meaning of symbols.
- Lesson 36. ORDERING of sticks relative to length is stressed in this lesson. Sometimes the elements are colored sticks only, sometimes the sticks have color and their number, sometimes the elements only have their length and name, sometimes the elements are FIGURES only.

- Lesson 37. Using the set of elements from 1 through 10, the students are asked to find subsets that satisfy certain conditions. Union and intersection of sets are means of imposing the conditions.
- Lesson 38. Here we create the number line by noting the point on a line that marks the end of a given element that is placed in a standard position, i.e. having the same starting point.
- Lesson 39. Here "trips along the number line" are introduced as elements for dyadic combination in lessons to follow.
- Lesson 40. The number line is extended to include the point 20 and direction is stressed relative to trips along the number line.
- Lesson 41. Student is asked to supply the missing element from the set of possible trips along the number line in order to make a given story true.
- Lesson 42. Students are given experience with the ASSOCIATIVE LAW OF ADDITION. Parentheses are introduced to indicate orders of operations.
- Lesson 43. Further work is done with the ASSOCIATIVE LAW OF ADDITION, including its application with three trips along the number line and in which the binary operation is defined to be one trip, followed by a second trip from the end point of the first trip.
- Lesson 44. Further work is done with the associative law using the elements, trips along the number line, and the binary operation defined in Lesson 43. ZERO and the Additive Inverse of a trip are experimented with.
- Lesson 45. More of Lesson 44.
- Lesson 46. COUNTING is introduced as a one-to-one mapping of elements of the set to be counted with the UNIT element of the student's set and THEN determining which element of the student's set is equivalent in length to the accumulation of these matched unit elements.
- Lesson 47. Counting is continued using Venn diagrams and emphasizing those subsets which later will be identified as Union, Intersection, Complement, etc.
- Lesson 48. Inequalities are introduced, including the symbols "greater than" and "less than." This lesson is primarily one of information.
- Lesson 49. The student is given experience with determining sets of elements that will make certain stories true and in which the stories are inequalities.
- Lesson 50. More work in finding solution sets to inequalities.
- Lesson 51. More work in finding solution sets to inequalities in which the elements are sometimes colored sticks, sometimes numbers, and sometimes trips along the number line.
- Lesson 52. Students are asked to place the proper symbol between two elements and thus gain experience with the law of trichotomy.
- Lesson 53. Further experience is given in the use of signs of equality and inequalities, in order to form true stories.
- Lesson 54. Students are asked to form true stories by placing equality and inequality signs between two much more complicated names of elements.

- Lesson 55. Applications of the law of trichotomy as applied to trips along the number line and the forming of new stories.
- Lesson 56. The binary operation of multiplication is defined upon the set of colored sticks.
- Lesson 57. Examples are shown depicting the multiplication of two elements from the set of colored sticks.
- Lesson 58. Further "funny-paper" sequences of pictures indicate the operations and their order required in multiplying two elements, colored sticks together.
- Lesson 59. The Student is asked to perform the operation of multiplication upon two colored sticks and to find the one element of his set that is equivalent to their product.
- Lesson 60. Students are asked to form true stories by inserting the proper element in place of the "x" in a story involving the binary operation of multiplication.
- Lesson 61. Dynamic patterns involving the commutative, symmetric, and transitive laws are placed in a sequence of some eighteen problems.
- Lesson 62. The symbol, " $\cdot$ ", is introduced to indicate the binary operation of multiplication, and more problems are given involving numbered colored sticks.
- Lesson 63. More problems in multiplication and true stories in which the elements are numbered sticks only.
- Lesson 64. Same as Lesson 63.
- Lesson 65. Same as Lesson 63.
- Lesson 66. Same as Lesson 63.
- Lesson 67. An evaluation of the knowledge involving multiplication is made.
- Lesson 68. Beginning of a complete evaluation of the total program to date.
- Lesson 69. The beginning of the building of a multiplication table.
- Lesson 70. Further work on multiplication table.
- Lesson 71. Further work on the multiplication table.
- Lesson 72. Beginning of the building of an addition table.
- Lesson 73. Further work on the addition table.
- Lesson 74. Manipulative experience necessary for problems involving the multiplicative inverse. Determining of the proper element to put in place of an "x" in order to form a true story in which the binary operation is multiplication.
- Lesson 75. Same as Lesson 74.

CHAPTER THREE

EXPERIMENTAL DESIGN AND PROCEDURES

Introduction

The purpose of this investigation was to find out to what extent pupils in the first grade are capable of learning, through programmed instruction, mathematics in terms of algebraic structures, as described in the previous chapter. In the experiment, 136 first-grade children were instructed through the use of teaching machines for a period of approximately fifteen weeks. At the end of this program, the children were tested on their understanding of basic mathematical principles through oral interviews and group tests. Their performance was compared with that of two control groups, who had received only conventional instruction in mathematics. One of these control groups was drawn from the first grade, the other from the fourth.

The regular instruction in arithmetic carried on at these schools in general followed text books adopted by the State Department of Education. For example, the first grade controls studied the book by Merton and Brueckner (1957), *Ready for Numbers*. Approximately thirty minutes a day was set aside for this instruction.

While "conventional instruction" is difficult to specify, the use of these control groups was felt to offer some information regarding what some children, at least, might learn with respect to mathematical principles. Seventy-eight children in the experimental group at School A, were given both the experimental program and the conventional work in arithmetic. The remaining experimental subjects, at School B, received only the experimental program and were given no arithmetical instruction by their classroom teachers.

Hypothesis

First-grade children who are taught mathematics, in terms of algebraic structures, by means of an automated teaching machine, for one term, will do better on an oral test of understanding and application of algebraic structures than (1) matched first-grade controls, and (2) fourth-grade pupils.

Subjects

The subjects for this experiment were drawn from three schools located in an upper middle class suburban area. The average intelligence quotient of the children in these schools is somewhat above average. The schools will be referred to as School A, School B, and School C for purposes of identification. Seventy-seven children in School A, the entire population of the three first grade classes in this school, were used as the experimental group. No classes in this school were used for control purposes. In School B, fifty-nine children, taken from two first grade classes, were used for the experimental group, while thirty-eight children in the first grade were drawn from two other classes as a control group. Fifty-six children who were in the first grade in the third school, School C, were used as additional control subjects. The children in the first grade had all received one semester of instruction. They were, therefore, completing the second half of their first year in school. Additional control subjects were drawn from the third and fourth grades at Schools A and B.

Controls

An excellent strategy for conducting an experiment of this type would be to select both the experimental group and the control group from the same class. This procedure, however, was not possible because of the administrative problems it would entail. Furthermore, the possibility of contamination through interactions among the children might be fairly high. Intact groups were therefore used.

Another desirable feature for this study would have been that of using a very large control group, sampled so widely that the results might be representative of what could be called conventional instruction. In this study, however, the three schools used were all in similar types of residential middle class neighborhoods. It was hoped in this way to minimize as far as possible the differences in background between the experimental and control subjects. For three of the comparisons an analysis of co-variance was used in an effort to control for initial differences. In analyzing results from some of the other tests, matched groups were constructed from the control and experimental populations.

The Teaching Machine Used

The teaching device used in this experiment was one developed by Hughes Aircraft Company and called the Videosonic Teaching Machine, Experimental Model. This instructional device presents a sequence of questions by projecting 2 x 2 kodachrome slides on the viewing plate (see Figure 3). Accompanying each projected picture is a commentary which is played to the subjects through earphones from a tape recorder within the machine. In front of the device is a response panel where the subject may register his choice of a correct answer by pressing one of three buttons. In this investigation each slide presented three choices on the screen. The child was to select and press the button in front of the correct picture. If the selection was correct, a green light came on beside the button, a new slide appeared, and the next question was asked. If the child selected the wrong answer, a yellow light appeared on the response panel. When the child pressed the button next to the yellow light, he cleared the machine and set it up again so that he could register another answer. The child was required to press the correct button before the machine would advance to the next slide.

Other Teaching Aids

An important part of the instructional program consisted of the use of various aids. Most important of these was the Cuisenaire Blocks (1954). The cross-section of each of these blocks is one centimeter square, but they vary in length from one centimeter to ten centimeters, in units of one centimeter. There are, therefore, ten different lengths of blocks, all the blocks of given length being of the same color.

In addition to the blocks, each child was supplied, as the program required, with a number of plastic symbols. These included "plus," "minus," "equals," "times," "greater than," and "less than" symbols. Later in the instructional program use was made of a number line, scaled for the Cuisenaire Blocks.

School A

Six machines were available for use at School A. These devices were placed in



FIGURE 3. Two of the six Teaching Machines in Use at School A

the only space available, a small room used for work with individuals or small groups. A seventh machine was available in this room in case a replacement was necessary. Six children arrived from their classes every twenty minutes during the day. A record was kept of the performance of each child by means of a Veeder counter, one counter for each device. At first these counters were placed at the research assistant's desk, and were reset to zero for each child. They registered the total number of errors each child made for each lesson or book. But during the first two weeks it was discovered that many children were guessing on items. In order to discourage this guessing behavior, the counter was placed beside the child and the circuits were adjusted so that the counters registered one point for each correct first attempt. If the child's first attempt was not correct, the counter would not register again for the same item even though on subsequent tries the right button was pressed. This procedure did have a marked effect on many children. They were found to wait longer before making their first response, and they seemed to pay more attention to the total scores they made each day. In fact, some competition developed between the children as demonstrated by the fact that they compared scores as they left the room.

Since only three copies of the program were available at this school, the children began the course in staggered fashion. Three children came the first day to the first program. The second day, six children came. On any given day, therefore, at least two and sometimes more lessons were being given children in the room.

Children who were absent were frequently given two programs during a single instructional period. It was also possible to have the children come in on special call during a period when some other child was absent. Occasionally children stayed during recess to make up a program which they had missed. Even so, a few children missed some of the daily lessons. Because only a limited number of programs could be accommodated in the experimental room, it was practically impossible to have every child take every program he had missed.

At School A no record was obtained for the individual errors made. But for the first two weeks at the end of each lesson the child's error score for that

lesson was recorded. The new scoring system with counters was installed during the third week. Total scores in terms of number correct were thus available from Lesson 22 on for each child. Since the error scores obtained prior to this point are not comparable to the items correct scores, they are not included in this report.

At School A occasional malfunctioning of the machine sometimes resulted in an inadequate presentation of the program. Most of the time when, for example, the tape was either advanced or retarded, children immediately spoke out to the research assistant. For many of the lessons, the same question was asked for an entire sequence of frames so that such dislocation did not make a serious error in presentation. But malfunctioning of the device did interfere with the presentation of the program on occasion. It was not always possible to have children take the lesson a second time.

School B

Since only two additional machines were available for the experiment at School B, children were taught in a special room by a single device in groups of eight to ten. The viewplate was removed from one of the devices and the image projected from the rear on a large translucent screen. Space was provided for ten children to view this projection simultaneously (see Figure 4). In front of each child was a table and screens were placed between the children to reduce social distractions. In front of each child was a panel consisting of three buttons, a red light, and a green light. Each of these panels was wired through a bank of relays to the teaching device. When a slide was presented, each child indicated on his own panel the answer he thought was correct. If his choice was correct, the green light went on and no other choices would register. If he made an error, a red light would go on, and now the child selected another answer until the green light indicated that he was successful. Each child heard the commentary through earphones connected with the teaching machine.

Although some homogeneity of learning ability was attempted by grouping students on the basis of pre-test scores, wide individual differences within each group

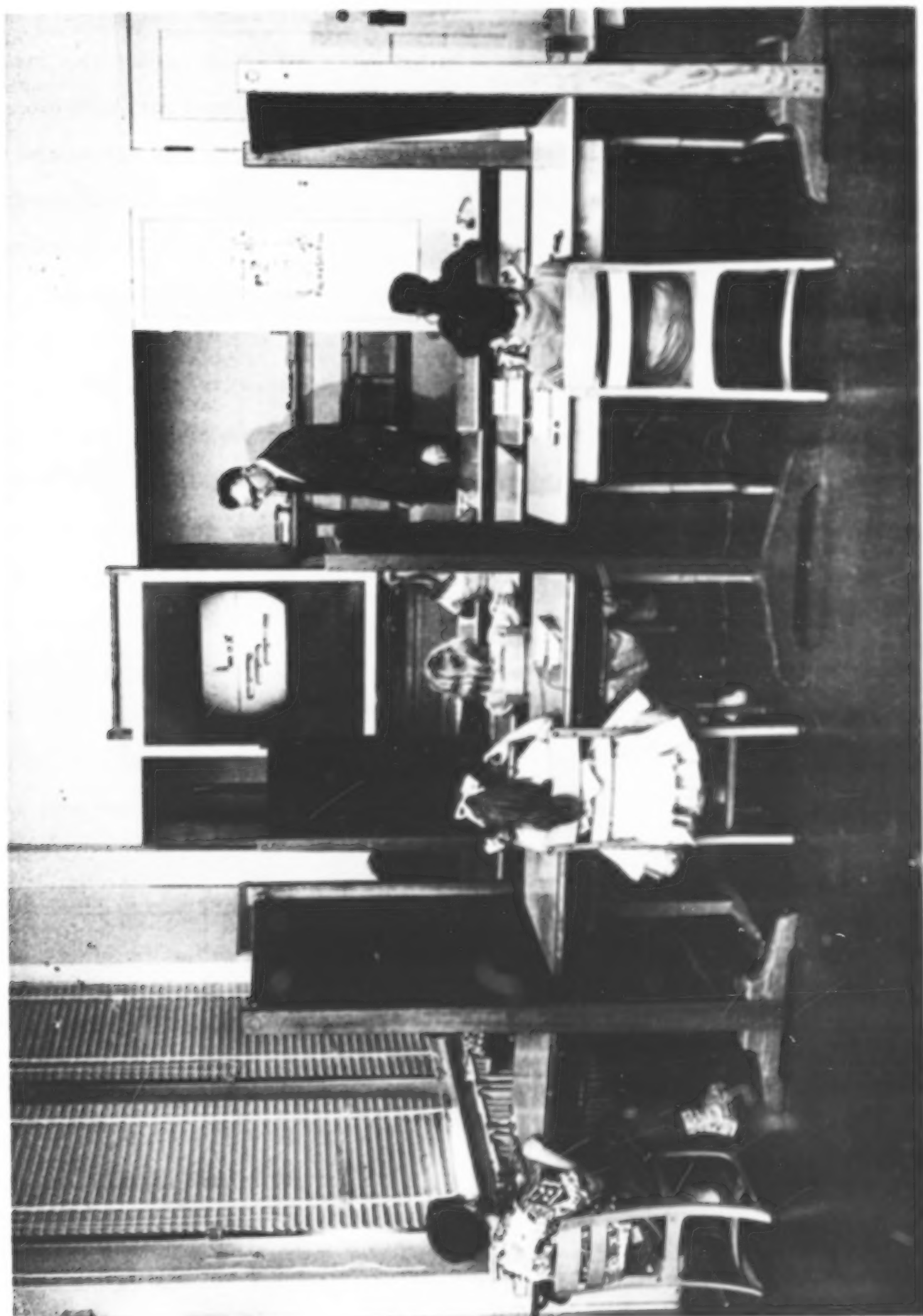


FIGURE 4. Group Teaching Machine in Use at School B
(Teaching Machine is behind translucent screen)

were still found. Some children took little time for each frame, while others required more time. Unlike the situation at School A where each child could move at his own pace, the pace of each of these groups of eight to ten children was limited to that of the slowest child. While children were waiting, therefore, there was a tendency to talk to each other and this resulted in the usual classroom discipline problem.

The average time taken to complete a lesson differed slightly among the groups. The range was from 12.1 minutes per day to 14.8 minutes per day, with the over-all average just about 13.8 minutes.

A recording procedure was developed for the responses of the children by using a paper tape on which perforations indicated the first-attempt errors each child made. This procedure was instituted at the beginning of the second week and, therefore, the results of the performance of the children at School B are available from the seventh lesson on.

The procedure used at School B meant that children came in a group of eight to ten, stayed exactly the same amount of time, and returned to their classes as a group. Teacher's reactions at School B seemed to be more favorable to this procedure than was true at School A where children came back to the room at irregular intervals.

The Performance of Children on the Program

As has been previously explained, adequate data was obtained at School A, beginning with the twenty-second lesson. At School B evidence was obtained from lesson seven on. The difficulty of the daily lessons was calculated through Lesson 70, although most of the children were given seventy-two lessons in all and a few even more. In interpreting the difficulty level of the program it should be noted that the program, except for the first few lessons was not revised; these figures therefore are for a first draft.

In computing the difficulty level of the lessons, the number of first-attempt failures was obtained for each child for each lesson. The average error rate for

each lesson for which records are available is presented in Table I; the same results are presented graphically in Figure 5. It is unfortunate that evidence regarding the difficulty of the first six lessons at School B and the first twenty-one lessons at School A cannot be provided. However, one may note that the difficulty levels at School A correspond closely with those at School B. Furthermore, the over-all average of the difficulty levels for all lessons, 22 through 70, is 34.2 per cent at School B and 33.6 per cent at School A. Such close agreement, obtained with completely different recording methods, suggests a fair degree of over-all accuracy even though there may have been individual exceptions. It may be assumed, therefore, that the difficulty levels of Lessons 7 through 21 at School A were not far from those reported for School B.

TABLE I

Average Difficulty Level of Each Book in Per Cent

Based on Seventy-Seven Children at School A
and Fifty-Four Children at School B

Number of Book	Number of Responses Frames in Book B	AVERAGE DIFFICULTY	
		Per Cent of Responses School A	Per Cent of Responses School B
7	18		18
8	14		25
9	6		25
10	16		30
11	16		18
12	11		24
13	18		22
14	18		24
15	18		24
16	15		30
17	15		27

TABLE I CONTINUED

Number of Book	Number of Response Frames in Book $\frac{a}{b}$	AVERAGE DIFFICULTY	
		Per Cent of Response Frames Missed School A $\frac{c}{d}$	School B $\frac{e}{f}$
18	14		30
19	18		25
20	18		17
21	18		19
22	18	19	21
23	18	11	15
24	18	16	22
25	14	14	16
26	18	34	34
27	17	20	24
28	18	21	27
29	17	17	30
30	17	28	36
31	18	13	12
32	18	31	37
33	18	32	36
34	18	38	40
35	17	29	30
36	16	44	35
37	16	37	29
38	11	36	27
39	18	46	49
40	15	42	38
41	18	48	48
42	17	35	32
43	15	46	53

TABLE I CONTINUED

Number of Book	Number of Response Frames in Book	APPROXIMATE DIFFICULTY	
		Per Cent of Response Frames Missed School A	School B
44	18	45	50
45	18	46	37
46	18	15	12
47	18	32	32
48	16	39	31
49	18	45	43
50	16	45	39
51	15	42	40
52	18	37	31
53	18	51	51
54	18	42	42
55	18	41	38
56	16	35	34
57	16	23	17
58	14	38	37
59	18	39	40
60	18	40	43
61	18	35	37
62	17	37	43
63	18	35	37
64	18	33	33
65	18	31	31
66	18	32	29

Number of Book	Number of Response Frames in Book ^{a/}	Average Difficulty	
		Per Cent of Response Frames Missed School A ^{b/}	School B ^{c/}
67	18	39	31
68	18	39	28
69	18	39	34
70	18	41	36

Total Average

Books 7 thru 70	1073		31.1
Books 22 thru 70	840	33.6	34.2

^{a/} In some of the books there were instructional frames which required no response by the pupil.

^{b/} Children at School A used individual teaching machines. These difficulty figures were based on the records obtained each day from each child's counter.

^{c/} Children at School B were taught by a group teaching machine. These figures were based on records obtained from the punched record tape.

As might be expected, the difficulty level of the individual lessons varies widely. The easiest lesson of those reported is Lesson 31, with an average difficulty level of 12 per cent, while the most difficult was Lesson 53, with a difficulty level of 51 per cent. With a few exceptions, the difficulty level of the earlier lessons appears to be somewhat less than the later ones.

The performance of children on this program also varied widely. For each child at School A, the average error rate was computed by tabulating the total number of first attempt failures for Lessons 22 through 70 and dividing by the number of frames attempted. The resulting index expressed in terms of per cent was recorded for each child. The resulting distribution of scores is presented in Figure 6. One child made as few as 15 per cent errors on these lessons, representing unusual mastery of the program content, while others made error scores of 50 per cent and higher.

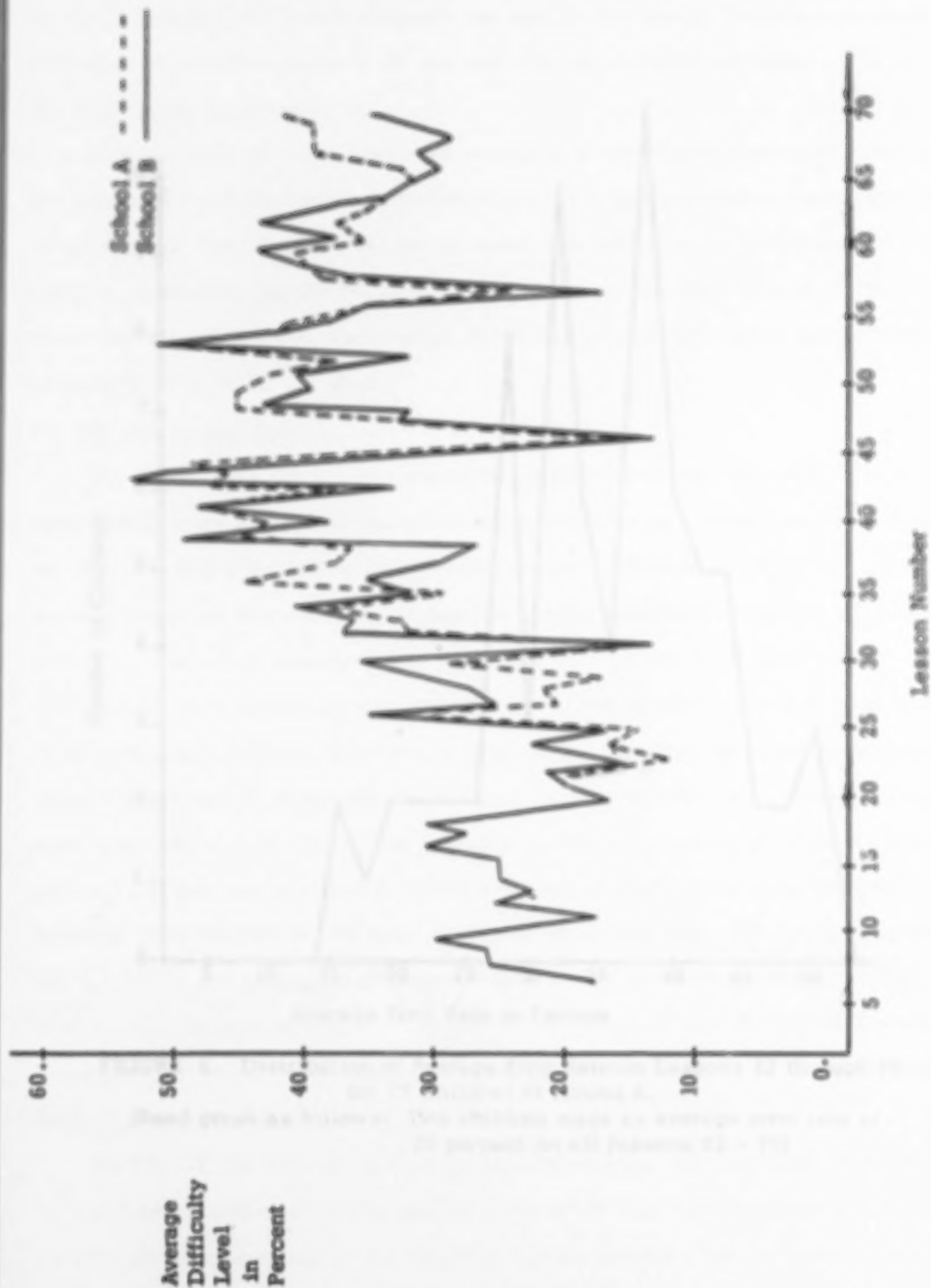


FIGURE 5. Average Difficulty Level for Lessons at School A and School B.

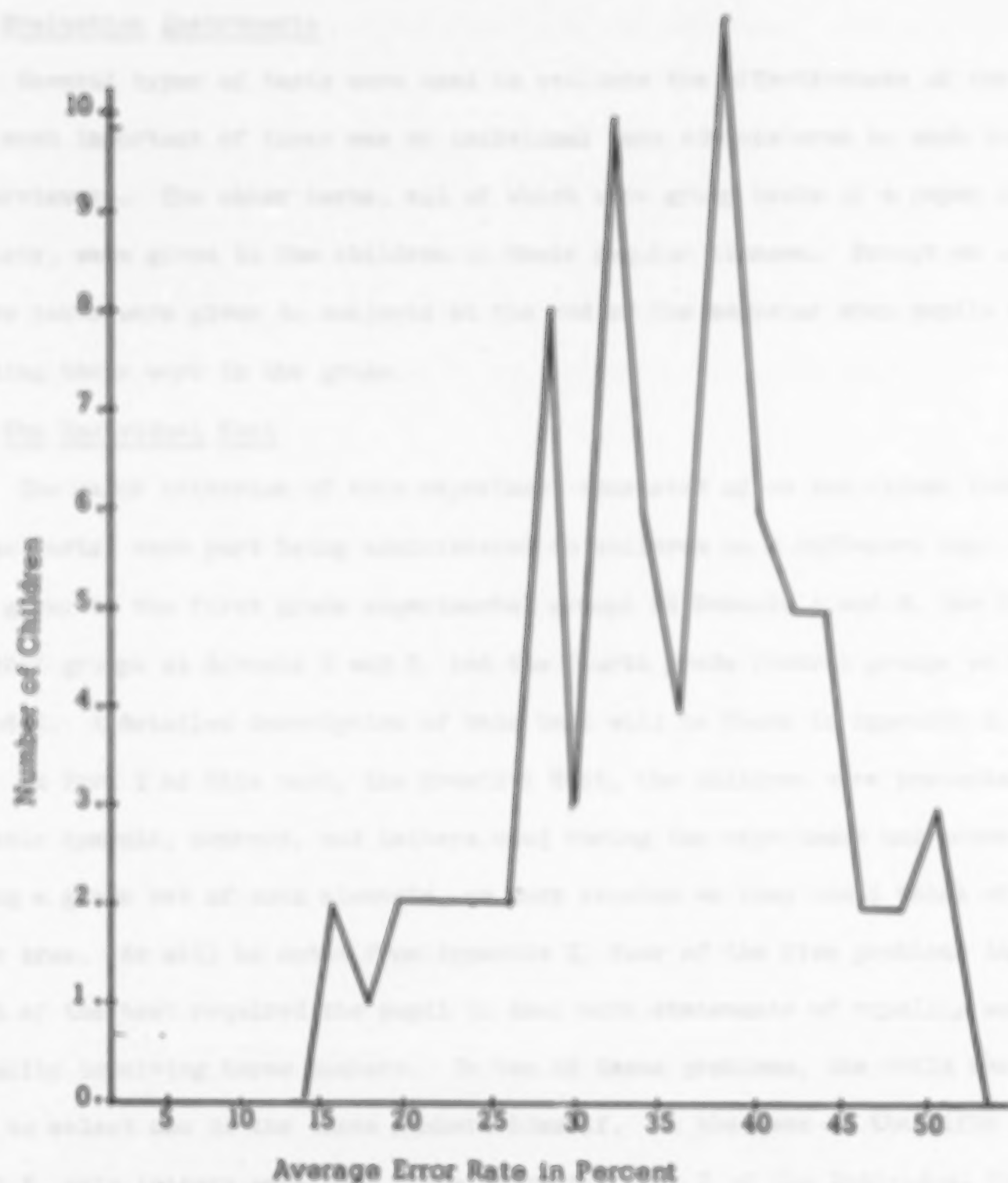


FIGURE 6. Distribution of Average Error Rate on Lessons 22 through 70 for 77 Children at School A.

(Read graph as follows: Two children made an average error rate of 20 percent on all lessons 22 - 70)

On these lessons, the latter children did very little better than chance, which is indicated by an error score of 67 per cent for three-choice problems.

The Evaluation Instruments

Several types of tests were used to evaluate the effectiveness of the program. The most important of these was an individual test administered to each child by interviewers. The other tests, all of which were group tests of a paper and pencil variety, were given to the children in their regular classes. Except as noted, these tests were given to subjects at the end of the semester when pupils were completing their work in the grade.

1. The Individual Test

The major criterion of this experiment consisted of an individual test in three parts, each part being administered to children on a different day. This test was given to the first grade experimental groups at Schools A and B, the first grade control groups at Schools B and C, and the fourth grade control groups at Schools A and C. A detailed description of this test will be found in Appendix I.

In Part I of this test, the Creative Test, the children were presented with plastic symbols, numbers, and letters used during the experiment and asked to make, using a given set of such elements, as many stories as they could think of which were true. As will be noted from Appendix I, four of the five problems in this part of the test required the pupil to deal with statements of equality and inequality involving three numbers. In two of these problems, the child was permitted to select one of the three numbers himself. In the case of the fifth test in Part I, only letters were used. The score for Part I of the Individual Test was the number of correct statements which the pupil could construct; no record was kept of wrong statements made by the pupils.

Part II of the Individual Test dealt with solving problems which were set up in the form of mathematical statements using these same plastic symbols. The first fifteen problems were all of the multiple choice variety. The last three involved the use of the number line chart.

Part III of the Individual Test consisted of application problems. Here an attempt was made to find out whether the experimental pupils were now better able to solve fairly practical problems from every day experience. For each problem there was a colored picture which was shown to the child as the interviewer read the instructions regarding the problem to the pupil. Most of these problems involved applications of the commutative, the associative, and the distributive law applied to addition. One of the problems involved the concept of zero. There were no time limits for these tests and interviewers were instructed to make sure that each child was as relaxed as possible and was trying to do his best.

2. Arithmetic Group Test

This test (reproduced in Appendix II) was constructed by Rod Shager as part of a study conducted for his doctoral dissertation in the Department of Psychology, University of California, Los Angeles, (1961). It consisted of three parts. The first, Test AR, involved ordering procedures. The child was shown two numbers separated by a space. He was asked to select one of the numbers, presented as alternatives at the side, which properly falls in the space between the first two. The second, Test NO, involved subtraction. In the box at the left were two numbers; the child was to select one of the four alternatives given on the right which signified the difference between the numbers in the box. The third, Test EO, dealt with the equivalence of sums. At the box at the left were two numbers. The child was to select the circle which contained those numbers whose sum would equal the sum of the numbers in the box at the left. The arithmetic group test was given to all of the first grade experimental and control subjects.

3. Group Transfer Test

A third evaluative instrument, called a Group Transfer Test (Appendix III), was designed by the principal investigator after some of the limitations in the individual interviews became apparent. It consisted of a test booklet in which each problem was presented on a separate page. Since each page was of a different color from the adjacent pages, all subjects were asked to answer each item at the same

time by being referred to the appropriate color. Children advanced to each new problem as a group.

The first six problems and the last two problems in this test required the pupil to select one of three alternatives as a substitute for "x" in the equation involving addition or subtraction. Three other problems dealt with inequalities.

The remaining five problems in this test required the pupil to rearrange the letters in an algebraic equation, so as to compose a "story" which would say the same thing in a different way. The letters T, H, and U were used exclusively so as to avoid any reversal problems in writing which might otherwise occur. These five problems involved the use of the principles of commutation, transposition, and symmetry. It was felt that an "understanding" of these principles could be assessed by requiring the subjects to generalize to a broader class after they had been given a limited example of the principle. Hence, each of the first three problems was preceded by an arithmetical example of the appropriate principle. Students were not asked to solve this example; instead they were shown the solution on the following page. For the problem itself, subjects were asked to write the letters in the appropriate places in the unfinished story given in the box.

This Group Transfer Test was an attempt to assess in more objective fashion some of the abilities which were being tapped by the individual test. The first six problems, for example, were identical to the first six problems of the problem-solving test, Part II of the Individual Test, except for the fact that children used paper and pencil instead of plastic symbols. Similarly, the problems involving the manipulation of letters were presented in an attempt to parallel Problem 5 of the Creative Test, Part I of the Individual Test. The last two problems in the Group Transfer Test were included so that the test would end with easy problems to maintain morale.

The Group Transfer Test was administered to all of the first grade experimental and control subjects at Schools A and B during the final days of the school year.

Time limitations prevented its administration at School C. However, this test was administered at the start of the following term to ninety-nine pupils beginning the third grade, and ninety-five pupils beginning the fourth grade in School B. It was hoped that some information might thus be obtained which would permit a comparison with older children, a comparison which was involved in the major hypothesis of the study.

4. Addition and Subtraction Test

A conventional addition and subtraction test was administered to the matched experimental and control subjects at School B to find out whether the lack of a conventional program in arithmetic handicapped the experimental group in this computational task. The test consisted of twenty-five problems requiring the child to add two single-digit figures, one placed beneath the other. The remaining twenty-five problems involved subtraction of single-digit numbers in the same spatial arrangement. Although this computational test was given to all subjects at School A, the results were not analyzed since this experimental group had been given the program as well as the conventional instruction in arithmetic.

The results of Part II of the Individual Test, the Standardized Test, are given in Table III and in Figure 8. Here, the experimental group with a mean of 11.4 is superior to the first grade control with a mean of 10.7. This difference is clearly significant at the .05 level. In this part of the Individual Test the fourth grade control had a mean score of 13.3. The difference between the mean of the experimental group and the fourth grade control in Part II was significant at the .05 level.

For Part III of the Individual Test, Applications, the results are given in Table IV and in Figure 9. No significant differences were found between the experimental group and the controls.

CHAPTER FOUR

RESULTS

The results obtained from each of the evaluation measures will be discussed in turn. An attempt will be made to draw the separate findings together and to interpret their significance.

The Individual Test

In Part I of the Individual Test, the Creative Test, the child's performance was scored simply in terms of the number of statements which the child composed; no points were subtracted for incomplete or erroneous statements. The results are given in Table II, where it will be seen that the experimental group's mean of 20.6 was practically as high as that of the fourth grade control group's 21.0; the difference is not significant. On the other hand, the first grade control group obtained a considerably lower mean score of 9.9. The difference between the first grade experimental group and the first grade control group was highly significant. These results are shown graphically in Figure 7.

The results of Part II of the Individual Test, the Problem-Solving Test, are given in Table III and in Figure 8. Here, the experimental group with a mean of 15.2 is superior to the first grade control with a mean of 12.2; this difference is clearly significant at the .01 level. On this part of the Individual Test the fourth grade control had a mean score of 19.5. The difference between the means of the experimental group and the fourth grade control on Part II was significant at the .01 level.

For Part III of the Individual Test, Applications, the results are given in Table IV and in Figure 9. No significant differences were found between the experimental group and the controls.

----- First Grade Experimental N = 128
 ----- First Grade Control N = 95
 ----- Fourth Grade Control N = 58

Table II. Results on Part I of Individual Post-Test: Control N = 58

The Creative Test

	Mean	S.D.
First Grade Experimental N = 128	20.6	10.8
First Grade Control N = 95	9.9	3.8
Fourth Grade Control N = 58	21.0	8.2

Difference between means of Exp. Group and 1st Grade Control = 10.70
 Standard Error of Difference = 1.03
 Significance of Difference: $P < .001$

Difference between means of Exp. Group and 4th Grade Control = .35
 Standard Error of Difference = 1.44
 Significance of Difference: not significant

Number of Cases



FIGURE 7. Distribution of Scores on Part I of Individual Post-test.
 The Creative Test

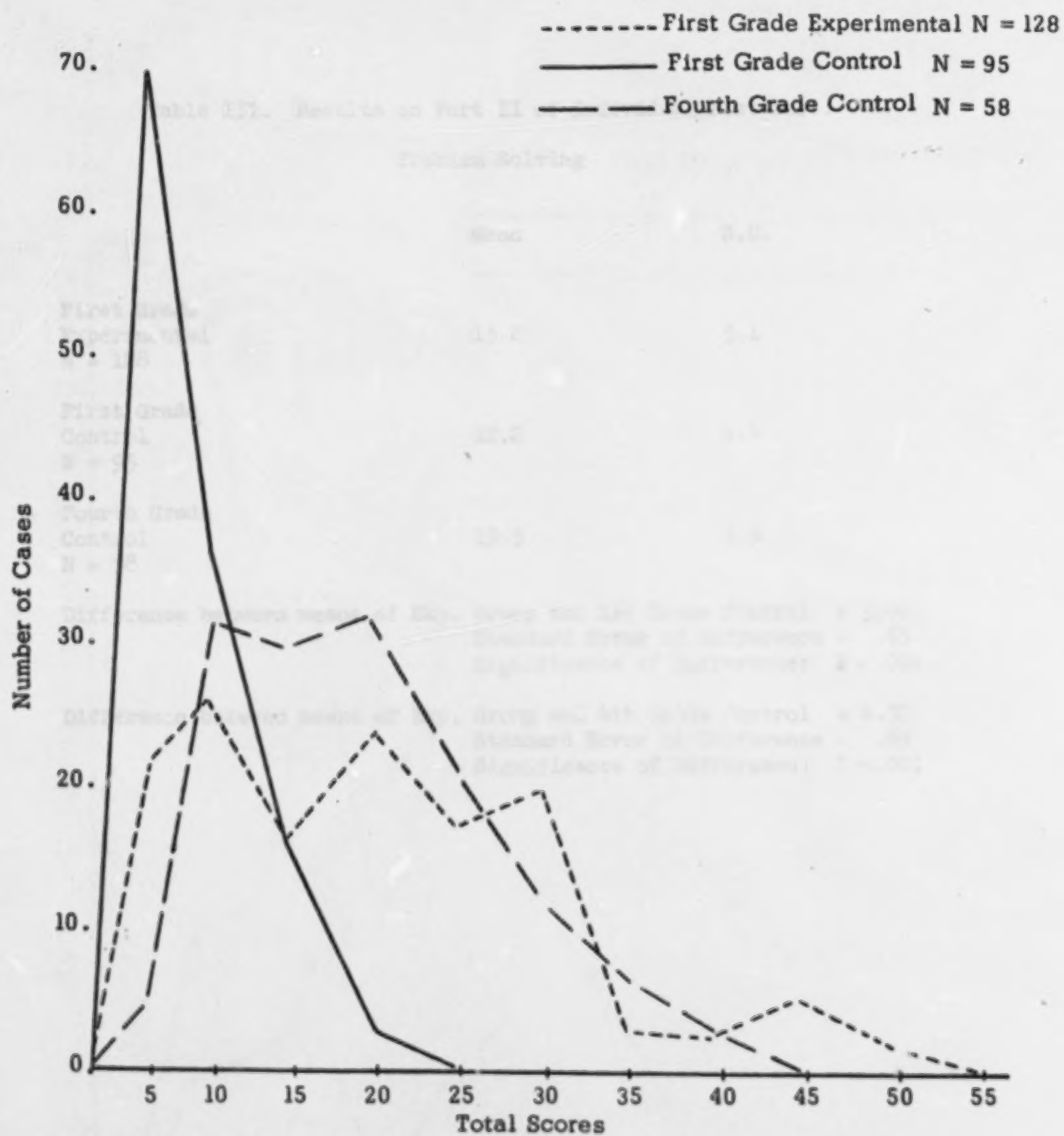


FIGURE 7. Distribution of Scores on Part I of Individual Post-test:
The Creative Test

Table III. Results on Part II of Individual Post-Test:

Problem Solving		First grade experimental N=128	First grade control N=95	Fourth grade control N=58
	Mean	S.D.		
First Grade Experimental N = 128	15.2	5.1		
First Grade Control N = 95	12.2	4.4		
Fourth Grade Control N = 58	19.5	3.9		

Difference between means of Exp. Group and 1st Grade Control = 3.00
 Standard Error of Difference = .65
 Significance of Difference: $P < .001$

Difference between means of Exp. Group and 4th Grade Control = 4.30
 Standard Error of Difference = .69
 Significance of Difference: $P < .001$

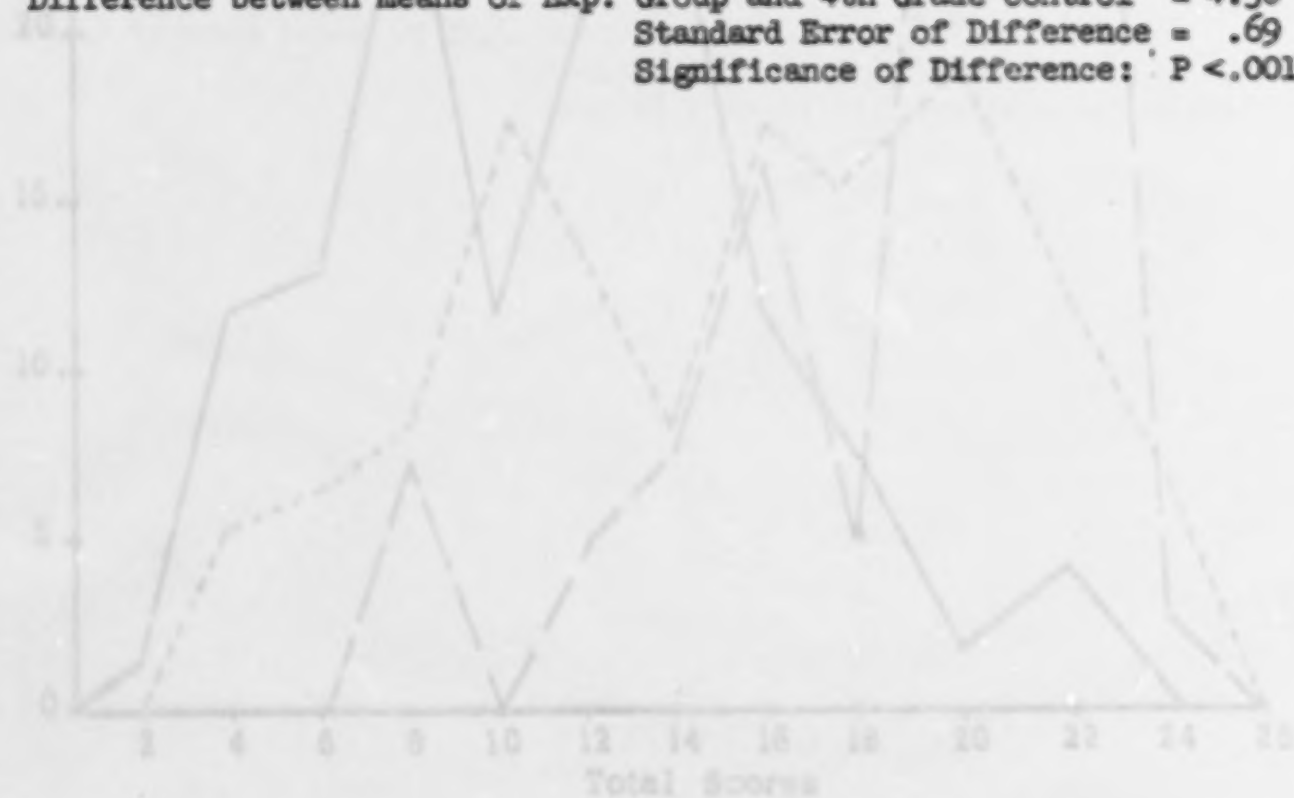


FIGURE 8 Distribution of Scores on Part II of Individual Post-Test:
 Problem Solving

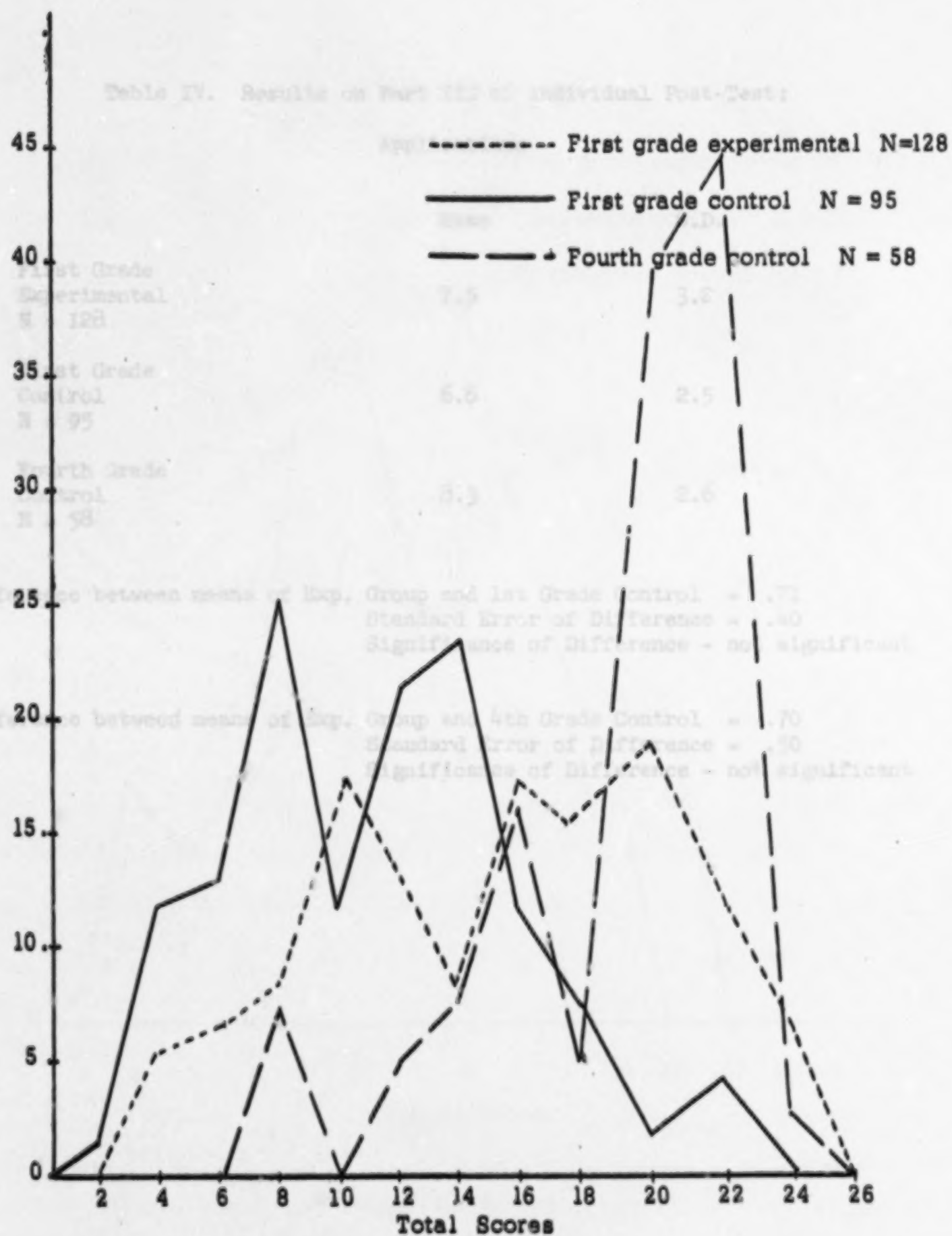


FIGURE 8 Distribution of Scores on Part II of Individual Post-Test:
Problem Solving

Table IV. Results on Part III of Individual Post-Test:

Applications			
	Mean	S.D.	N
First Grade Experimental N = 128	7.5	3.2	128
First Grade Control N = 95	6.8	2.5	95
Fourth Grade Control N = 58	8.3	2.6	58

Difference between means of Exp. Group and 1st Grade Control = .71
 Standard Error of Difference = .40
 Significance of Difference - not significant

Difference between means of Exp. Group and 4th Grade Control = .70
 Standard Error of Difference = .50
 Significance of Difference - not significant



FIGURE 9 - Distribution of Scores on Part III of Individual Post-Test Applications

The results of the total test are given in Table 7 and in Figure 10.

Table V. Results on Total Individual Post-Test

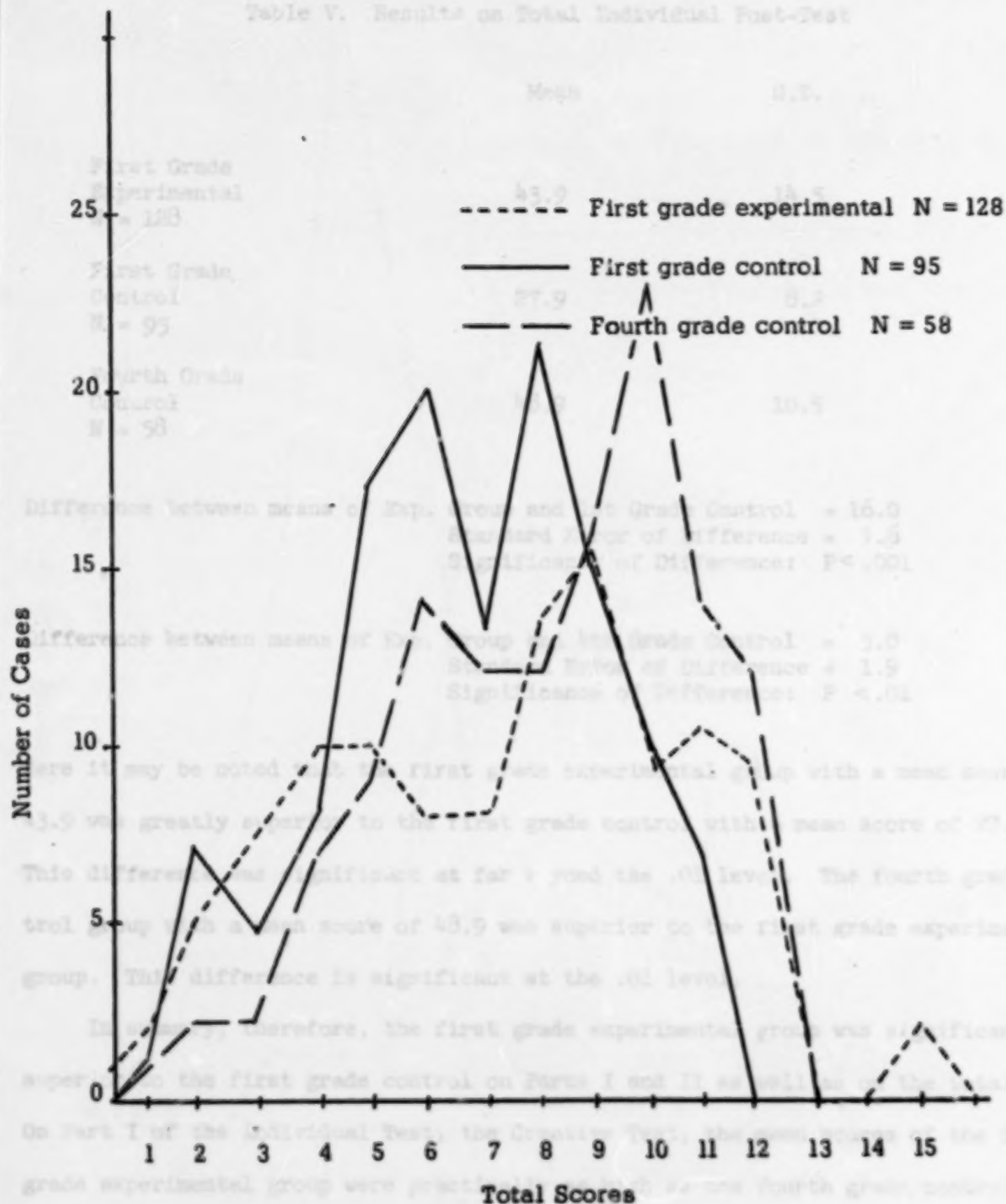


FIGURE 9 Distribution of Scores on Part III of Individual Post-Test: Applications

Group Arithmetic Test

The Group Arithmetic Test was administered to the first grade experimental and

The results of the total test are given in Table V and in Figure 10.

Table V. Results on Total Individual Post-Test

	Mean	S.D.
First Grade Experimental N = 128	43.9	14.5
First Grade Control N = 95	27.9	8.2
Fourth Grade Control N = 58	48.9	10.5

Difference between means of Exp. Group and 1st Grade Control = 16.0
Standard Error of Difference = 1.6
Significance of Difference: $P < .001$

Difference between means of Exp. Group and 4th Grade Control = 5.0
Standard Error of Difference = 1.9
Significance of Difference: $P < .01$

Here it may be noted that the first grade experimental group with a mean score of 43.9 was greatly superior to the first grade control with a mean score of 27.9. This difference was significant at far beyond the .01 level. The fourth grade control group with a mean score of 48.9 was superior to the first grade experimental group. This difference is significant at the .01 level.

In summary, therefore, the first grade experimental group was significantly superior to the first grade control on Parts I and II as well as on the total test. On Part I of the Individual Test, the Creative Test, the mean scores of the first grade experimental group were practically as high as the fourth grade control. Only on Part II, Problem Solving, and on the total test was there a significant difference in favor of the fourth grade control over the first grade experimental group.

Group Arithmetic Test

The Group Arithmetic Test was administered to the first grade experimental and

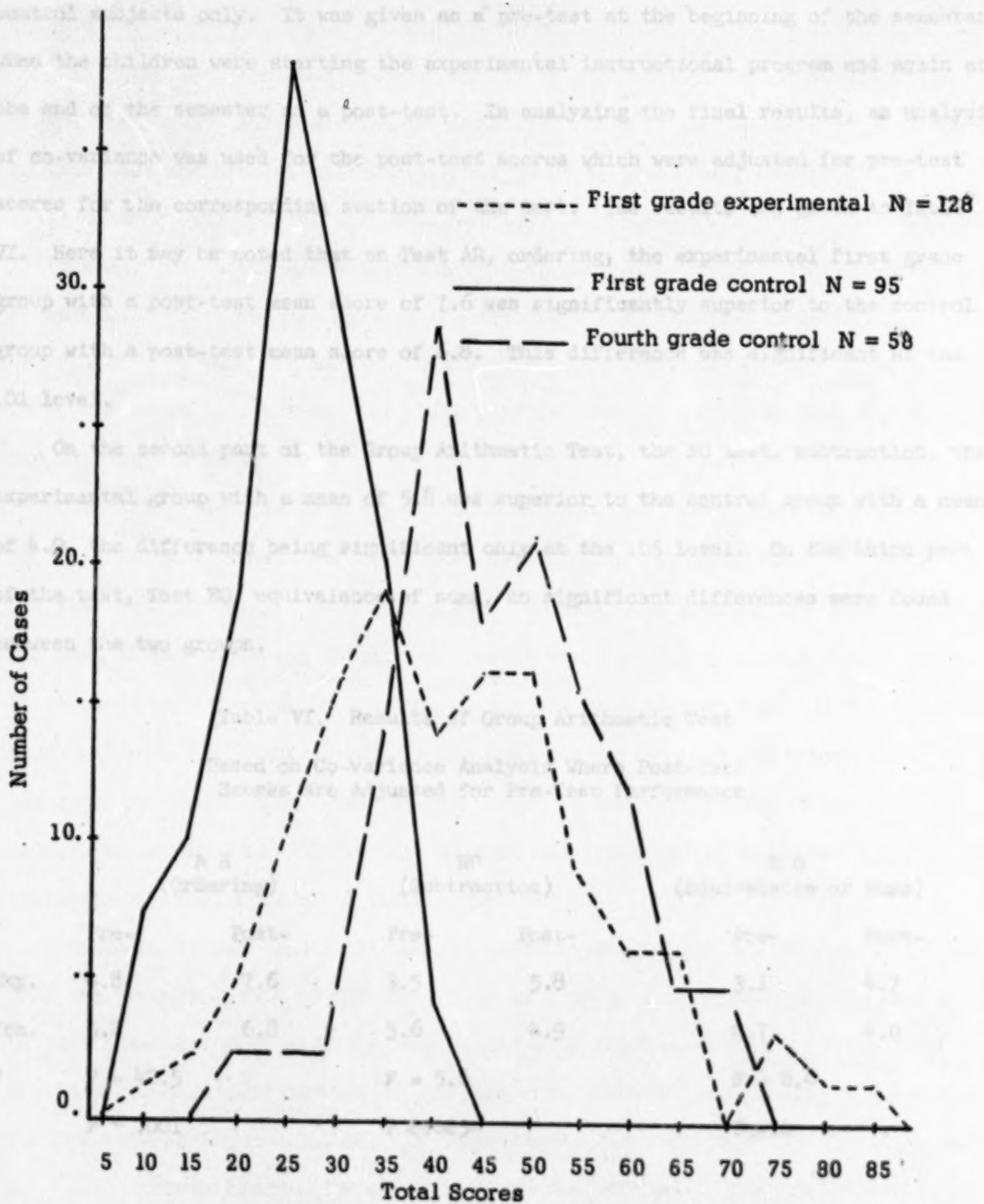


FIGURE 10 Distribution of Scores on Total Individual Post-Test

control subjects only. It was given as a pre-test at the beginning of the semester when the children were starting the experimental instructional program and again at the end of the semester as a post-test. In analyzing the final results, an analysis of co-variance was used for the post-test scores which were adjusted for pre-test scores for the corresponding section of the test. The results are given in Table VI. Here it may be noted that on Test AR, ordering, the experimental first grade group with a post-test mean score of 7.6 was significantly superior to the control group with a post-test mean score of 6.8. This difference was significant at the .01 level.

On the second part of the Group Arithmetic Test, the NO test, subtraction, the experimental group with a mean of 5.8 was superior to the control group with a mean of 4.9, the difference being significant only at the .05 level. On the third part of the test, Test EO, equivalence of sums, no significant differences were found between the two groups.

Table VI. Results of Group Arithmetic Test

Based on Co-variance Analysis Where Post-Test Scores are Adjusted for Pre-Test Performance.

	A R (Ordering)		NO (Subtraction)		E O (Equivalence of Sums)	
	Pre-	Post-	Pre-	Post-	Pre-	Post-
Exp.	4.8	7.6	3.5	5.8	3.1	4.7
Con.	5.2	6.8	3.6	4.9	2.7	4.0
F	F = 42.5		F = 5.8		F = 2.4	
	P < .001		P < .025		P > .1	

Group Transfer Test

Although this test was not designed to be used as a major criterion of the study, the results are presented in Table VII. Here it may be noted that on Part I of this test the mean of the experimental group is lower than that of the first grade control, but only at the .05 level of significance ($t = 1.97$). But both the third grade and fourth grade controls were significantly higher than the first grade experimental group on this part of the test ($t = 7.29$ and 8.40 respectively). Tests of significance were not made on Part 2 since this consisted of only three items. It may be noted that the mean of the experimental group is higher than the mean of the first grade control.

On Part 3, dealing with algebraic statements, the first grade experimental group achieved a higher mean score than did the first grade control, although the difference is not significant ($t = 1.65$). On this same part of the test, Part 3, the mean of the beginning third grade pupils was significantly higher than the first grade experimental ($t = 3.39$), as was the mean of the beginning fourth grade pupils ($t = 5.77$). From these data one cannot conclude that the experimental group was superior to the first grade control group on the test involving rearranging letters in an equation. On the other hand, the experimental group was clearly inferior to the beginning third grade group.

Since an inspection of differences on individual items may be useful, in Table VIII are presented data on the percentage of each of four groups passing each item. Of particular interest here are comparisons with respect to Part 3 of the test. It may be noted that only on item 10, dealing with commutation, is there a large difference between the control and the experimental first grade groups ($p_1 = .56$ and $p_2 = .76$). The difference is significant at the .05 level ($t = 2.33$) using a test of the significance of difference between two proportions.

TABLE VII
Results of Group Transfer Test

	First Grade Experimental N = 119 Mean S.D.	First Grade Control N = 36 Mean S.D.	Beginning Third Grade N = 99 Mean S.D.	Beginning Fourth Grade N = 95 Mean S.D.
Part 1				
Addition and Subtraction Problems Items 1-6, 15-16	6.27 1.62	6.83 .93	7.56 .74	7.74 .60
Part 2				
Inequalities Items 7-9	2.34 .71	2.00 .82	2.64 .63	2.76 .45
Part 3				
Algebraic Statements Items 10-14	1.89 .96	1.58 1.06	2.36 1.05	2.64 .90

TABLE VIII
Per Cent Passing Each Item on Group Transfer Test
for First Grade Experimental and Control and for Beginning Third and Fourth Grade Controls

Item Number	First Grade Experimental N = 117 Schools A & B	First Grade Control N = 36 School B	Beginning Third Grade N = 101 School B	Beginning Fourth Grade N = 96 School B
Part 1 Addition and Subtraction				
1	85	92	99	99
2	84	97	100	99
3	87	97	99	97
4	82	100	91	95
5	78	94	95	100
6	49	42	89	98
15	83	86	96	97
16	74	75	86	96
Part 2 Inequations				
7	85	89	95	96
8	71	47	79	92
9	79	64	90	91
Part 3 Algebraic				
10	76	56	92	92
11	50	42	71	69
12	41	44	48	58
13	17	8	23	47
14	9	8	3	4

While one may conclude that the experimental group was able to generalize with respect to commutation better than the control, it is of interest to note the performance of the two groups on item 14 which required pupils to rewrite the equation $T - U = H$. This item was extremely difficult for all subjects, including the third and fourth graders. But 50% of the first grade experimental subjects commuted by writing $U - T = H$ while only 22% of the control did so. This difference in proportion is significant at the .01 level ($t = 2.97$). In this respect the first grade experimental were similar to the third and fourth grade--a larger proportion of whom also commuted on this problem (54 and 57 per cent respectively). These results suggest that the first graders comprehended commutation enough to use it, like older children, in the wrong problem. The control group didn't know enough about the principle to misapply it.

Addition and Subtraction

The arithmetic and subtraction test described earlier was administered to all subjects in the experimental and control groups at School B. Of particular interest here is the fact that experimental subjects in this school were given no instruction in arithmetic other than what was contained in the experimental program. (It will be recalled that the experimental subjects at School A were given the conventional course in arithmetic as well as the auto-instructional program.) Although the instructional program did not specifically aim for competence in computational skills, it was of interest to determine whether the experimental group suffered in this respect. Two matched groups were formed from the experimental and control subjects at School B, each group consisting of 36 subjects. The pairs were matched on the basis of Skager's Arithmetic Group Test given as a pre-test at the beginning of the term. On the Addition and Subtraction Test, with a possible total of 50 points, the mean of the experimental group was 47.1 with a standard deviation of 4.5. The mean of the control group was 47.6 with a standard deviation of 6.8. The difference was not significant ($t = .35$). While the distributions were highly skewed and the use of a non-parametric test is indicated, it seemed obvious that negative differences would be found and further analysis was not carried out. There was no evidence

from this test, therefore, that the experimental group was inferior in computational skills at the end of its course of instruction.

Observational Data on Pupil Interest

Informal observations of the pupils indicated that the experimental subjects were generally quite eager to appear each day for the auto-instructional lessons. Although records made by the research assistants contain a few isolated instances of boredom, there was little evidence of the "pall effect" sometimes found with programmed instructional materials. Mr. Crawford reports as follows:

Contrary to the opinions of the present meaningful traditional educational psychologists a program involving nothing more than patterns of meaning-free abstract symbols can hold the attention of first grade pupils for at least fifteen weeks, twenty minutes a day. Not only can it hold their attention, it captures their interest and imagination to the point that many of them refused to miss their mathematics lesson for assemblies, play days, or many other extra-curricular activities. Many expressed unhappiness at the conclusion of the program when told it was not to be continued. Many parents indicated that the program was a chief topic of discussion at the dinner table and that many students performed unassigned homework.

The intrinsic reinforcement gained through visual and tactile senses proved to be far superior to extrinsic reinforcement supplied by machine with green light and verbal confirmations. This was clearly indicated by the expressions on the pupils' faces at the time when they discovered the solutions with their sticks and the anticlimatic reaction to the reinforcements of the machine.

Discussion

Any comparison between experimental and control classes is always difficult. In this investigation, for example, five first grade classes drawn from two schools were used as a source for control subjects. It is not possible to ascertain to what extent these five classes are representative. Even more serious is the fact that it is not at all clear as to what kind of instruction was being given the control groups. Although the control subjects were in general following a course of study as represented by the text and workbook, the effect of each teacher far outweighs that of the instructional material used. It seems safe to conclude, however, that

the control subjects were given no formal instruction with respect to algebraic structures as incorporated in the experimental program. It seems also clear that the control groups were given no formal instruction in inequations; they were not specifically taught to deal with "greater than" and "smaller than" types of relationships. The investigation does not constitute therefore a test of one form of instruction versus another; the control groups were used to determine what the experimental subjects might be expected to know about these relatively advanced concepts in the absence of specific instruction of such topics.

1. Arithmetic Type Skills

The general results of the various instruments reported appear to support the position that the experimental group was not handicapped with respect to conventional skills in arithmetic. The results of the Arithmetic Group Test prepared by Rod Skager demonstrate that the experimental group was not inferior to the control groups in addition and subtraction when original performance on these skills was held constant. While the results of the Group Transfer Test indicated some superiority of the control classes on addition and subtraction, these data were unreliable. Also, they were taken from only two classes and did not represent the broader sampling obtained by Skager's test. It should be remembered, however, that Skager's analysis included all the subjects in the experimental group--the majority of whom attended School A where the conventional course in arithmetic was given along with the experimental program. From his results it is not clear as to whether the children in the experimental group did better because of a combination of the conventional course in arithmetic and the instructional program or because of the effects of the program alone.

The results from the addition and subtraction test at School B with matched control and experimental groups are more appropriate for an evaluation of the program in this respect. Here it is apparent that there is no evidence to suggest that the subjects who were instructed only by means of the experimental program were inferior to the controls with respect to computational skills.

It seems fairly clear that the experimental group was superior to the first grade control group in dealing with ordering of numbers. On the first part of Skager's test significant differences were obtained. The experimental group was also superior on Part 2 of the Group Transfer Test, although these differences were not tested for significance. Since the conventional instruction does not usually include such an emphasis, it is reasonable to assume that the program was successful in helping children learn about relationships of "greater than" and "smaller than".

2. Understanding of Algebraic Structures: The Individual Test.

Arithmetical competence was not the major objective of the program. The major hypothesis of this investigation dealt with the development of an understanding of algebraic structures and for this purpose the central criterion was the Individual Test. The results from this instrument indicate that the performance of first grade children in the experimental group were significantly superior to those of the control group on both the Creative Test and the test of Problem Solving. These results are subject to several criticisms. First, was there a possibility of interviewer bias? Second, what was the effect of not recording pupils' mistakes; and third, to what extent did this test measure familiarity with test materials, and ability that was not related to the hypothesis?

a. Standardization of Test Procedures

A problem relating to any individually administered test is whether the interview procedures are sufficiently standardized so as to permit comparisons of student performance as determined by different interviewers. If there is evidence that different interviewers using the same instrument yield different results, the reliability and validity of the test must be questioned. The standardization of test administration is effected by the extent to which interviewers follow the same specific procedure in both (a) presenting problems to the child and (b) in scoring and recording the child's performance.

Although a critical study of the instructions given to the interviewers (Appendix I) is useful, empirical evidence is available to determine whether the different interviewers actually used the instrument in a standardized fashion. In

this study seven interviewers were used to test the subjects in the first grade experimental group. If the interviewers give reliably different mean scores to the samples (drawn at random) from the experimental population, the standardization procedures may be questioned. Of course, the converse does not hold; the samples may not differ reliably but there still may be a lack of standardization.

A record of the name of the interviewers was kept only for Part I of the Individual test, the Creative Test. For Parts II and III, which were administered on different days, a record was not made of who interviewed each child. Using the data obtained only from Part I, therefore, a test was made to determine to what extent interviewers gave different results for the random samples of children they interviewed from the experimental group. The results are presented graphically in Figure 11, where the distribution of scores assigned by each interviewer is presented. It may be seen here that there was considerable variation among the means for these seven samples of experimental subjects. They range from 12 for Interviewer G to 35 for Interviewer C.

To find out whether these differences could be attributed to random sampling of the subjects or whether the interviewers reliably differed in the way they administered the test and scored the child's performance, an analysis of the variance was carried out using the two schools as one factor and the interviewers as a second. For differences among the interviewers, a significant F of 26.3 ($N_1 = 4$, $N_2 = 98$) was obtained. This is significant far beyond the .001 level. No significant differences were found between the schools. The interaction term was also insignificant. Since there was a lack of homogeneity among the distributions, a non-parametric test was used. Employing the Kruskal-Wallis one-way analysis of variance, a corrected H of 59.6 was obtained. This figure was also highly significant.

We may conclude, therefore, that the interviewers differed reliably in the way they administered and scored Part I of the Individual Test. (Evidence of this sort on Parts II and III is not available.) Such lack of standardization is highly understandable considering the nature of the test. Every effort was made in the

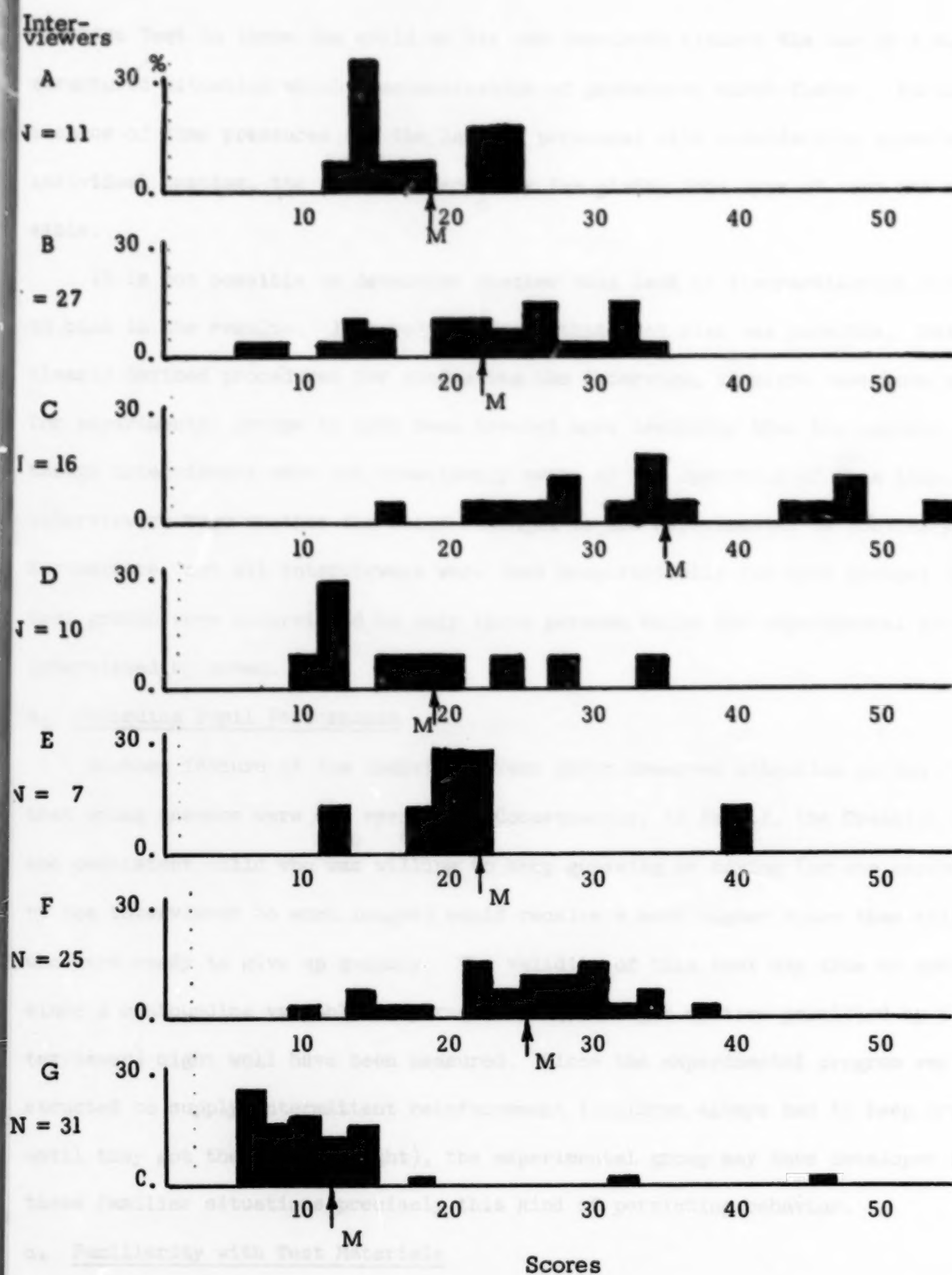


FIGURE 11. Distributions of Scores for Experimental Group Obtained by Different Interviewers for Part I of the Creative Test.

Creative Test to throw the child on his own resources without the use of a highly structured situation which standardization of procedures might foster. Furthermore, because of time pressures and the lack of personnel with considerable experience in individual testing, the training necessary for giving this type of test was not possible.

It is not possible to determine whether this lack of standardization contributed to bias in the results. It merely suggests that such bias was possible. Without clearly defined procedures for conducting the interview, it might have been possible for experimental groups to have been treated more leniently than the control even though interviewers were not consciously aware of the operation of this bias. The interviewers knew whether the child belonged to the experimental or control groups. Furthermore, not all interviewers were used proportionally for both groups; the control groups were interviewed by only three persons while the experimental groups were interviewed by seven.

a. Recording Pupil Performance

Another feature of the Individual Test which deserves attention is the fact that wrong answers were not recorded. Consequently, in Part I, the Creative Test, the persistent child who was willing to keep guessing or trying (or one encouraged by the interviewer to work longer) would receive a much higher score than children who were ready to give up quickly. The validity of this test may thus be questioned since a confounding variable of persistence (or length of time permitted by the interviewer) might well have been measured. Since the experimental program was constructed to supply intermittent reinforcement (children always had to keep trying until they got the problem right), the experimental group may have developed for these familiar situations precisely this kind of persisting behavior.

c. Familiarity with Test Materials

In comparing the experimental with the control group, it is also necessary to recognize that the particular materials used were highly familiar to the trained experimental pupils. To some extent, of course, this familiarity was precisely

what the program was trying to develop. On the other hand, it is clear that the initial instructions given the students (See Appendix I) were hardly adequate to help the control group learn to deal with essential aspects of the situation which were in no way related to the basic content of mathematics.

For example, the children in the experimental group had been taught the word "story" to represent mathematical statements in solving problems throughout the 15 weeks of the semester. The preliminary instructions given to each child did give one illustration of what was meant by a "story," but this was probably quite inadequate to help these young children to get accustomed to such a different use of a very familiar word. Furthermore, the use of plastic symbols instead of written ones to represent mathematical signs presented the control group with an unfamiliar situation. The symbols "greater than" and "less than" were undoubtedly completely new to the control group, as was the use of "." to indicate multiplication. While they may have understood the relationships involved and could have expressed them in more familiar words, they may have been handicapped by the necessity of translating into this new way of talking. What is implied here is that the results of the Creative Test may have been quite different had the control students been given a short period of instruction, say only 20 minutes in length, in order to familiarize themselves with the terminology and symbols used in the experiment. The experimental group, therefore, may have performed relatively well simply because they had become acquainted with features of the testing situation that were somewhat extraneous to the problem of measuring the understanding of basic mathematical principles.

As a test of this hypothesis of handicap because of the test situation, the first six items on the mimeographed Group Transfer Test were used. These six items are identical with the first six items on Part II, Problem Solving, of the Individual Test: the only difference is the mode of presentation. Three matched groups were formed by matching, on the basis of the pre-test in arithmetic given by Rod Skager, each of 34 pupils in the control group at School B with a pupil in the experimental group at School A. The scores for each of these pupils were computed on (1) the

first six items of Part II of the Individual Test and (2) the first six items of the Group Transfer Test. The handicap score was obtained for each pupil by subtracting (1) from (2). The mean handicap score for the control group at School B was 2.18 as compared with a mean handicap of .29 for the experimental group at School B and a mean of .91 for the experimental group at School B and a mean of .91 for the experimental group at School A. On the basis of the difference method (using matched pairs) the difference between the mean of the control group and each of the two experimental groups was significant ($t = 5.6$ for the smallest comparison). Such a result should of course be interpreted in terms of possible lack of standardization in the administration of the Group Transfer Test, since the test was given by two administrators to intact groups. Although instructions were read to the pupils, there is always the possibility of variation. In any case, the lack of agreement between these two measures suggests that the assessment procedures used in this investigation need considerable attention.

3. The Problem of the Criterion

This investigation has pointed up the importance of the criterion problem. For the measurement of arithmetic skills of computation, ordering, and problem solving, satisfactory items appear to have been developed. But the assessment of "understanding of algebraic structures" has been inadequate in this investigation. Part I of the Individual Test, the Creative Test, possesses excellent features. The child is called upon to use his own resources in "creating" statements with a minimum of cues. But it is not always clear on what basis he arrives at his final answers. For the first four problems children could use arithmetic facts alone in constructing statements. If this were the case the test would then be essentially a measure of his ability to recall arithmetical facts involving the three numbers used for the problem set. If, on the other hand, a child faced with one statement " $3 + 5 = 8$ " immediately constructs another of the form " $5 + 3 = 8$," one might with greater assurance infer that the child was using the commutative property of numbers whether he was able to verbalize it or not. To arrive at inferences of this type it is necessary to record the responses of the child in sequential order.

In problem 5 of the Creative Test the child was required to generalize to a situation where his arithmetic facts would not help him. Here it seems very clear that the child would be able to create true statements (unless he were arranging the letters at random) only if he understood the postulates involved. His knowledge of the arithmetic facts do not help him here. This approach was also adopted in Part 3 of the Group Transfer Test. While this latter test possessed the advantage of supplying information as to which postulate the child was using, it may have the weakness of providing too much direction for the subject. The distinct advantage of the Creative Test in this respect is that the child may select his own postulates as he chooses. The mediational activity required is therefore more complex and probably involves the type of complex thinking which is desired as an outcome.

Another feature of Part 2 of the Group Transfer Test involved the child's ability to generalize from one class to a new one. Here, for instance, an arithmetical example of the commutative property of numbers was supplied and the child was asked to extend the principle to a non-arithmetic statement. Although this process of generalizing to new classes is important, it is possible that many children were able to obtain the correct solution to the problem by mechanically manipulating the letters in the same way as they had manipulated the numbers. Children who solved the problem in this mechanical fashion are undoubtedly displaying only a superficial understanding. This was probably reflected in the final item in Part 3 where half of the experimental subjects and those in grades three and four commuted for problem $T - U = H$.

Mr. Crawford believes that "understanding is better measured by an observation of the process, than by the seemingly objective scoring of the end product of the process." He points out that

This ability to create is best demonstrated in the examples in which first grade pupils were given a choice of third elements. (Part I, Problems 3 and 4). For example, control group pupils invariably chose the element six to go with the given elements two and four when asked to form as many true relations as possible using the three elements. In contrast, the experimental pupils chose nine or zero. The control group would form the relationship $2 + 4 = 6$ and stop. The experimental

group would usually start with the inequation $2 + 4 < 9$ and rapidly proceed to form other true relationships such as $9 + 2 > 4$, and $9 + 4 > 2$, etc. This seems to indicate that the experimental students have a better understanding of relationships between elements than the control students who merely play back that which was given to them by their teachers.

If understanding is to be defined in terms of process, considerable attention needs to be given to the recording of the subject's behavior during the Individual Test so that an assessment of the reliability of the data-gathering process may be made. Such a record should probably include the sequence of statements he constructs, the choices of elements he makes, smiles and comments accompanying each constructed statement, and so forth. There is a host of literature and methodology available on the topic of using these recording methods in data-gathering.

The construction of an instrument to assess "understanding of algebraic structures" poses many difficult problems. It seems probable that a wide variety of different measures should be developed. Some of these would provide specific information regarding which postulate a child is using. Others might offer objective data on the child's ability to solve problems under standardized conditions, or his ability to think structurally about mathematics where he must deal with many postulates at a single time. Still others might provide a record of the sequence of steps and other behavior demonstrated by a child when presented with problem situations.

4. Instructional Time

It may be of some value to point out that there was a difference between the control and experimental group at School B in terms of total time spent in instruction. But the results of this investigation do not permit the drawing of any conclusions regarding relative efficiency, even though this is an important factor to consider in educational decisions. Some of these difficulties have been referred to at the beginning of this chapter.

The experimental subjects at School B spent an average of less than 14 minutes per day. Since approximately the last fifth of the program was devoted to multiplication, the experimental subjects spent less than 12 weeks on the topics of addition and subtraction. The control subjects spent 30 minutes a day for over 17 weeks.

CHAPTER FIVE

CONCLUSIONS

These results demonstrate that in terms of computational skills in addition and subtraction the experimental subjects learned no less than the control subjects --in two classes at least--in much less than half the time.

But the post-tests were not designed to assess all of the kinds of learning which the control subjects may have acquired. These children may have learned to be more skillful at solving practical problems or in other ways not relevant to the experiment. It seems fairly certain that although there was no significant difference in computation skills in addition and subtraction, a more extensive and precise evaluation of these pupils would reveal that the two groups had learned different things. Which set of outcomes is more important to teach is a matter for additional study.

The experimental subjects, who were in general somewhat above average in ability, missed an average of 31 per cent of the items of the program, excluding the first six. Since each frame had three choices, 67 per cent would indicate sheer guessing. Most investigators who insist on low error rates would consider this figure quite good for an untrained first draft of a program, especially when one considers that the operation of the machine and the administrative procedures may have added extraneous sources of difficulty. They would also maintain that the relatively high error rate would indicate the need for considerable revision.

For a program of this type, error rate does not necessarily have to be low. It is possible that children may make many errors before they discover the solution to a class of problems. Further revision of the program should include a study of to what extent children were simply guessing on the frames, and to what extent they might have been "trying out hypotheses." The latter could be assessed if children suddenly began to get the correct answers to all members of a series of problems involving the same concept or principle.

Until a careful revision of this program is undertaken, better materials, instruments created, and administrative procedures improved, it may be undesirable to arrive at conclusions regarding the intellectual abilities of third grade children.

CHAPTER FIVE

CONCLUSIONS

An important but frequently overlooked phase of research is the initial period of hypothesis development, for, without fruitful hypotheses, research no matter how well executed lacks significance. The investigation reported herein probably should be viewed primarily as hypotheses formulation activity. The instructional program is a bold and imaginative one; it possesses considerable promise as a child's first introduction to the field of mathematics and it suggests new directions for subsequent investigations in mathematics education.

However, the program in its present form, except for the first few daily lessons, is in need of revision. The experimental subjects, who were in general somewhat above average in ability, missed an average of 31 per cent of the frames of the program, excluding the first six. Since each frame had three choices, 67 per cent would indicate sheer guessing. Most investigators who insist on low error rates would consider this figure quite good for an unrevised first draft of a program, especially when one considers that the operation of the machines and the administrative procedures may have added extraneous sources of difficulty. They would also maintain that the relatively high error rate would indicate the need for considerable revision.

For a program of this type, error rate does not necessarily have to be low. It is possible that children may make many errors before they discover the solution to a class of problems. Further revision of the program should include a study of to what extent children were simply guessing on the frames, and to what extent they might have been "trying out hypotheses." The latter could be assumed if children suddenly began to get the correct answers to all members of a series of problems involving the same concept or principle.

Until a careful revision of this program is undertaken, better criterion instruments created, and administrative procedures improved, it may be undesirable to arrive at conclusions regarding the intellectual abilities of first grade children.

Premature evaluation of promising ideas may lead, on the one hand, to uncritical acceptance and false leads or, on the other hand, to unwarranted rejection. If Edison's first electric light had been compared with then-existing methods of illumination, it would have come out a poor second-best.

With this thought in mind, a review and discussion of the findings of the investigation will be undertaken:

1. Arithmetic Skills

Although the development of skills in arithmetic was not a central aspect of the investigation, it is of interest to note that the experimental group who received only the instructional program were not inferior to controls in addition and subtraction when matched on the basis of pre-test scores. This was true even though the instructional program took less than half the instructional time as compared with the conventional arithmetic instruction. Undoubtedly, control and experimental subjects learned far more than what was indicated by these measures. What they learned is, however, undoubtedly quite different, so that no comparison of efficiency of instruction is possible.

There is evidence of the success of the program in teaching children to order numbers and to deal with "greater than" and "smaller than" relationships. This topic was presented as part of the program and is not generally included in conventional instruction in arithmetic.

2. Interest of Experimental Subjects in the Program

On the basis of informal observations, one may conclude that the children found the program appealing. While a few instances of apathy were noted, most children were eager to come each day. There was some evidence of the fact that children were interested enough in the content to discuss it with their parents. Informal observations by Mr. Crawford revealed that frequently children found great satisfaction in discovering the solution to a class of problems, a reward which appeared to outweigh the indication of success supplied by the device.

3. Learning of Algebraic Structures

The evidence as to how much the experimental group learned about algebraic structures is dependent upon the validity of the major evaluative instrument used for the experiment, the Individual Test. The results of this test clearly indicate that the experimental first grade pupils were superior to the first-grade controls in terms of their understanding of algebraic structures. On the first part of this test, the Creative Test, the mean of the experimental group of first grade children was almost equal to that of fourth grade children, and far above the mean of the first grade controls. It was this part of the test which attempted to assess the child's ability to use patterns of postulates in the construction of mathematical statements. On the second part of this test dealing with Problem Solving, the first grade experimental group were significantly superior to first grade control subjects but significantly below the level set by the fourth grade controls. No significant differences were found among these three groups on the third part of the Individual Test dealing with Applications.

In considering these results certain possible limitations of the Individual Test must be kept in mind:

(a) The different interviewers gave significantly different sets of scores to the children they interviewed from the experimental group. This suggests a lack of standardization of the interview procedures and the possibility of interviewer bias. Most of the interviewers used for testing the experimental groups were not used for the controls.

(b) The materials and some of the symbols and terminology used (e.g. the word "story") were entirely new to the control group. It is doubtful that they could have learned very much about the symbols as a result of the introduction provided. Some symbols were never explained in the test instructions. It is impossible to determine how much of the superiority of the experimental group was due to their familiarity with the materials and their knowledge of terminology, features of the testing situation which were extraneous to the problem of measuring an understanding

of algebraic structures.

There is some evidence that the control subjects were handicapped on the Individual Test. In comparison with matched experimental subjects, control children did significantly worse on a sample of items in the Individual Test than when these same items were presented subsequently in mimeographed form. It is not clear, however, that the mimeographed test was as good as might be desired.

(c) Since wrong answers were not recorded or used in the scoring of Part I of the Individual Test, a persistent child (or one given extra time or encouragement by the interviewer) could continue through trial and error to make up statements and thus increase his score. No time limits were involved. Since the experimental group had encountered difficult lessons (on a few lessons the average error rate was as high as 50 per cent), many of these children may have acquired precisely this type of persistence under these familiar conditions.

These considerations do not demonstrate that the individual test was invalid; they merely lessen our confidence in drawing conclusions from such data. Certainly, as Mr. Crawford points out, "if the creative part of the post-test is valid, it is clear that first grade pupils have an ability to create new relationships between elements when given a true relationship for these elements." And if one accepts the validity of the test, one must conclude with Mr. Crawford:

That first grade pupils have the ability to find solution sets to equations and inequations presented with abstract mathematical symbols is indicated by the fact that the experimental pupils averaged fourteen correct solutions out of twenty-four possible. This was 25% better than the first grade control group and 80% as well as the fourth grade control group.

Mr. Crawford summarizes the results of the findings as follows:

The primary purpose of this explanatory research was to attempt to discover if first grade pupils have the ability to understand mathematics when it is presented in a "meaning-free" structural form. That they do, in fact, have this ability is clearly demonstrated by the fact that the one hundred thirty-six experimental first

graders averaged 14.2 problems right, out of every eighteen problems in each daily lesson.¹ The standard deviation for the experimental group for this same distribution was only 1.8. Thus all first graders in the experimental group demonstrated a satisfactory ability to comprehend mathematics when presented in this structural form.

Without careful specification as to what the problems were, a low error rate cannot be taken as evidence of comprehension. And if the individual test is strongly affected by the way in which interviewers administered the test, by the scoring system employed, and by the factor of familiarity with the test materials, it is difficult to arrive at any conclusions regarding the extent to which the experimental children had acquired an understanding of mathematics structure.

4. The Need for Better Measures.

This study has brought into focus the importance of the criterion. If conclusions are to be drawn from a research study, it is essential that some empirical type of assessment be obtained. For this purpose a variety of instruments may need to be developed to assess the extent to which children have acquired a structural approach to mathematics. These should probably include measures which can supply information as to (1) which particular postulate a child is using, (2) the extent to which the child can generalize to new classes of problems, (3) how well the child can deal with a pattern of relationships in the presence of minimal cues, (4) the sequence of steps he follows in dealing with problems, and (5) the general behavior of the child in facing situations calling for mathematical activity.

But we act imprudently if we are in a hurry to reach conclusions and improperly attempt to measure behavioral changes which are slow in developing. It may take years of experience in a curriculum before subtle changes (e.g. certain aspects of the way a child thinks) become susceptible to measurement. For this reason we need long-range experimental programs that last for several years. In these cases we

¹ This difficulty level of 19 per cent was based on certain lessons in the first part of the program.

cannot make conclusions about possible long-range effects until such studies are carried out; we may be completely wrong about the anticipated outcomes.

5. Summary Statement

In summary, we may conclude that the auto-instructional program used in this investigation was successful in developing certain competencies in arithmetic, such as ordering of numbers, with no apparent loss in ability to handle simple addition and subtraction. And the instructional content and procedures appeared to be of considerable interest to the children.

A conclusion regarding the major hypothesis of this investigation, the extent to which first grade children are able to learn mathematics in terms of algebraic structures, hinges on one's judgment of the validity of the Individual Test used as the major criterion. If its validity is accepted, the results dramatically favored the hypothesis that children are able to understand mathematics in this fashion. A more cautious interpretation suggests that the program had an influence on the way these children thought through mathematical problems, but the extent and the nature of the effect are not clear. In any case, this challenging approach to the mathematics education of first grade children in terms of algebraic structures deserves thorough study and long-range experimentation.

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ALGEBRAIC STRUCTURES for FIRST GRADE

Part I: The Creative Test

Directions for the interviewer:

Materials: In each of the five problems in the creative test, the pupil being tested is to be given the following materials:

1. the three numbers peculiar to that problem,
2. one "plus" sign,
3. one "minus" sign,
4. one "equals" sign,
5. one "greater than" sign,
6. one "less than" sign.

Price: The interviewer is to say this and only this as he presents the above elements to each pupil:

1. "Here are three numbers, , , and ."
2. "Here is a 'plus' sign."
3. "Here is **APPENDIX I**"
4. "Here is **The Individual Test**"
5. "Here is **on**"
6. "Here is a 'less than' sign."

on

"You are to build as many true stories as you can using all of the three numbers in **Algebraic Structures for First Grade** all of the other signs in each story. For example,

by

— **Natalie Wilson and Robert Crawford** —

is a true story. Can you think of any other true stories that you could build using , , and once and only once in each story?"

If the pupil succeeds in building a true story, the interviewer should then say, "That's right. Now can you build another different true story?"

When the pupil admits by his actions or with words that he can't think of any other true stories with these particular three numbers, the interviewer will total his score for that particular problem and move on to the next problem.

Scoring: The pupil is to be given one point for each true story that he is able to build that satisfies the conditions imposed for that particular problem. The interviewer will find a scoring sheet with all possible correct solutions for each problem in the material furnished him. Thus, each correct solution can be easily checked as the student finishes it.

ALGEBRAIC STRUCTURES for FIRST GRADE

Part I: The Creative Test

Directions for the interviewer:

Materials: In each of the five problems in the creative test, the pupil being tested is to be given the following materials:

1. the three numbers peculiar to that problem,
2. one "plus" sign,
3. one "minus" sign,
4. one "equals" sign,
5. one "greater than" sign,
6. one "less than" sign.

Voice: The interviewer is to say this and only this as he presents the above elements to each pupil:

1. "Here are three numbers, __, __, and __."
2. "Here is a 'plus' sign."
3. "Here is a 'minus' sign."
4. "Here is an 'equals' sign."
5. "Here is a 'greater than' sign."
6. "Here is a 'less than' sign."

"You are to build as many true stories as you can using all of the three numbers in each story. You do not need to use all of the other signs in each story. For example,

$$\_ + \_ = \_$$

is a true story. Can you think of any other true stories that you could build using __, __, and __ once and only once in each story?"

If the pupil succeeds in building a true story, the interviewer should then say, "That's right. Now can you build another different true story?"

When the pupil admits by his actions or with words that he can't think of any other true stories with these particular three numbers, the interviewer will total his score for that particular problem and move on to the next problem.

Scoring: The pupil is to be given one point for each true story that he is able to build that satisfies the conditions imposed for that particular problem. The interviewer will find a scoring sheet with all possible correct solutions for each problem in the material furnished him. Thus, each correct solution can be merely checked as the student finishes it.

ALGEBRAIC STRUCTURES

Part I: The Creative Test

Problem 1. Interviewer should give pupil numbers 2, 4, and 6.

Possible
Answers

$$2 + 4 = 6$$

$$4 + 2 = 6$$

$$6 = 2 + 4$$

$$6 = 4 + 2$$

$$2 = 6 - 4$$

$$6 - 2 = 4$$

$$4 = 6 - 2$$

$$6 + 2 > 4$$

$$2 + 6 > 4$$

$$4 < 6 + 2$$

$$4 < 2 + 6$$

$$6 + 4 > 2$$

$$4 + 6 > 2$$

$$2 < 6 + 4$$

$$2 < 4 + 6$$

$$4 - 2 < 6$$

$$6 > 4 - 2$$

Example: $2 + 4 = 6$. Interviewer should then ask pupil to build as many other stories that are true as he can using all the numbers and symbols (signs) in the sample story.

When a pupil admits by his actions or with words that he can't think of any more stories with these particular numbers and symbols (signs), interviewer should replace the "plus" sign with a "minus" sign and again ask pupil to build more stories using 2, 4, 6, -, and = signs that are true.

When pupil admits by his actions or with words that he can't think of any more true stories with these particular numbers and signs, interviewer should replace equals sign with "greater than" sign and repeat above procedure.

Finally, "greater than" sign is replaced by "less than" sign and pupil asked to proceed as above.

Interviewer is to check proper place opposite the particular true story that the pupil creates. THIS IS VERY IMPORTANT for statistical study depends on interviewer checking proper slot.

After this problem is completed, interviewer should total number of true stories created by pupil with the given numbers and signs and proceed onto the next problem.

ALGEBRAIC STRUCTURES
Part I: The Creative Test
(Continued)

Problem 2. Interviewer is to give pupil the numbers 0, 1, and 2.

Possible
Answers

Example: $1 + 0 < 2$

$$1 + 0 < 2$$

$$0 + 1 < 2$$

$$2 > 1 + 0$$

$$2 > 0 + 1$$

$$2 + 0 > 1$$

$$0 + 2 > 1$$

$$1 < 0 + 2$$

$$1 < 2 + 0$$

$$1 + 2 > 0$$

$$2 + 1 > 0$$

$$0 < 1 + 2$$

$$0 < 2 + 1$$

$$2 - 1 > 0$$

$$0 < 2 - 1$$

$$1 - 0 < 2$$

$$2 > 1 - 0$$

$$2 - 0 > 1$$

$$1 < 2 - 0$$

Interviewer should then ask pupil to build as many true stories as he can using only the numbers and signs in the sample problem.

When the pupil admits by his actions or his words that he can not think of any more true stories with these particular numbers and signs, interviewer should replace less than sign with greater than sign and again request pupil to build as many true stories as he can using all the numbers and signs.

When a pupil admits by his actions or his words that he can not think of any more true stories with these particular numbers and signs, interviewer should replace "+" sign with "-" sign and repeat as before.

Interviewer is to check the proper place opposite the particular true story that the pupil creates. THIS IS VERY IMPORTANT for the statistical study depends on the interviewer checking the proper slot.

After this problem is completed, interviewer should total the number of true stories created by the pupil with the given numbers and signs and proceed on to the next problem.

ALGEBRAIC STRUCTURES
Part I: The Creative Test
(Continued)

Problem 3. Interviewer is to give pupil the numbers 1 and 3 and pupil is to be allowed to choose a third number from the set of numbers (4,5,6,7,8,9).

Possible
Answers

$$1 + 3 \leq X$$

$$3 + 1 \leq X$$

$$X \geq 1 + 3$$

$$X \geq 3 + 1$$

$$X - 3 \geq 1$$

$$1 \leq X - 3$$

$$X - 1 \geq 3$$

$$3 \leq X - 1$$

$$X + 1 > 3$$

$$1 + X > 3$$

$$3 < X + 1$$

$$3 < 1 + X$$

$$X + 3 > 1$$

$$1 < X + 3$$

$$1 < 3 + X$$

$$3 - 1 < X$$

$$X > 3 - 1$$

Example: $1 + 3 \leq X$

*Interviewer will use and accept the "equals" sign if the pupil chooses the number "4", otherwise the interviewer will use the inequality sign and accept as possible answers only those with the proper inequality sign.

Interviewer should then ask the pupil to build as many true stories as he can using the numbers and signs available to him. All signs, "=", "+", "-", "<", ">", should be available but interviewer should replace a sign at a time as in previous two problems until pupil admits by his actions or words that he can not think of any more true stories using the given numbers and signs.

Interviewer should check in the proper place each true story that the pupil creates. THIS IS VERY IMPORTANT FOR STATISTICAL REASONS. A total for the problem should be noted before proceeding on to the next problem.

ALGEBRAIC STRUCTURES
Part 1: The Creative Test
(Continued)

Problem 4. Interviewer is to give pupil the numbers 4 and 7 and allow him to choose a third number from the set (0,1,2,3).

Possible
Answers

$$X + 4 \leq 7$$

$$4 + X \leq 7$$

$$7 \geq X + 4$$

$$7 \geq 4 + X$$

$$7 - 4 \geq X$$

$$X \leq 7 - 4$$

$$7 - X \geq 4$$

$$4 \leq 7 - X$$

$$7 + X > 4$$

$$X + 7 > 4$$

$$4 < 7 + X$$

$$4 < X + 7$$

$$7 + 4 > X$$

$$4 + 7 > X$$

$$X < 7 + 4$$

$$X < 4 + 7$$

$$4 - X < 7$$

$$7 > 4 - X$$

Example: $\underline{X} + 4 \leq 7$

*Interviewer is to use and accept the "=" sign if pupil chooses "3", otherwise, the inequality sign.

As before, interviewer is to ask pupil to build as many true stories as he can using the given numbers and signs. The interviewer should replace the signs as in the previous problems.

Interviewer is to mark the proper slot for each true story that the pupil creates. After pupil by his actions or his words indicates that he can not think of any more true stories, interviewer should total the number of true stories created for this problem and proceed on to the next problem.

ALGEBRAIC STRUCTURES
Part 1: The Creative Test
(Continued)

Problem 5. Numbers are A, B, and C.

**Possible
Answers**

$$A + B = C$$

$$B + A = C$$

$$C = A + B$$

$$C = B + A$$

$$C - B = A$$

$$A = C - B$$

$$C - A = B$$

$$B = C - A$$

$$C + A > B$$

$$A + C > B$$

$$B < C + A$$

$$B < A + C$$

$$C + B > A$$

$$B + C > A$$

$$A < C + B$$

$$A < B + C$$

$$B - A < C$$

$$C > B - A$$

Interviewer is to tell the pupil that

$A + B = C$ and $A < B$ are true stories.

Interviewer should actually set up the story $A + B = C$ and then ask pupil if he can create another true story using the numbers A, B, and C AND the two signs "+" and "=".

When the pupil by his actions or words admits that he can not think of any other true stories using the particular numbers and symbols (signs), the interviewer should replace one of the signs and repeat his instructions about creating true stories. Each true story should be credited in the proper slot.

When the problem is completed, interviewer should total the number of true stories created.

ALGEBRAIC STRUCTURES
Part II: Solving Problems

Directions: In the first fifteen (15) problems in this part, the interviewer should first set up the problem (equation or inequation) and then he should set up the set from which a solution is to be chosen. After the problem has been set, the interviewer should read the commentary associated with that particular problem. The number of points to be scored for each problem will be indicated. The correct answer is underlined.

$$3 + 5 = X$$

$$7 \quad \underline{8} \quad 9$$

$$4 + X = 7$$

$$1 \quad 5 \quad \underline{3}$$

$$X + 5 = 9$$

$$\underline{4} \quad 6 \quad 22$$

$$7 - 4 = X$$

$$\underline{3} \quad 11 \quad 4$$

$$10 - X = 6$$

$$2 \quad \underline{4} \quad 5$$

$$X - 7 = 2$$

$$8 \quad \underline{9} \quad 10$$

$$X + X > 7$$

$$3 \quad \underline{4} \quad \underline{5}$$

$$X + X < 5$$

$$\underline{0} \quad \underline{1} \quad \underline{2}$$

1. Which of the numbers in the set (7, 8, 9) should be put in place of the X in the story to make the story true? (One point for the correct answer, 8.)
2. Which of the numbers in the set (1, 5, 3) should be put in place of the X in the story to make the story true? (One point for the correct answer, 3.)
3. Which of the numbers in the set (4, 6, 2) should be put in place of the X in the story to make the story true? (One point for the correct answer, 4.)
4. Which of the numbers in the set (3, 11, 4) should be put in place of the X in the story to make the story true? (One point for the correct answer, 3.)
5. Which of the numbers in the set (2, 4, 5) should be put in place of the X in the story to make the story true? (One point for the correct answer, 4.)
6. Which of the numbers in the set (8, 9, 10) should be put in place of the X in the story to make the story true? (One point for the correct answer, 9.)
7. Which of the numbers in the set (3, 4, 5) should be put in place of the X's in the story to make the story true? (Two points; one for the answer, 4, and one for the answer, 5.)
8. Which of the numbers from the set (0, 1, 2) should be put in place of the X's in the story to make the story true? (Three points; one for each answer, 0, 1, 2.)

ALGEBRAIC STRUCTURES
Part II: Solving Problems
 (Continued)

$$X - X < 2$$

$$4 \quad 5 \quad 10$$

$$2 \div 3 + 4 = X$$

$$5 + 4$$

$$2 + 7$$

$$\underline{5 + 7}$$

$$10 - (4 - 2) = X$$

$$\frac{(10 - 4) - 2}{(10 - 4) + 2} = X$$

$$10 - 2$$

$$2 \cdot (1 + 3) = X$$

$$2 \cdot 4$$

$$\frac{2 + 3}{2 + 6}$$

$$2 \cdot (4 + 1) = X$$

$$\frac{(2 \cdot 4) + 1}{(2 \cdot 4) + (2 \cdot 1)} = X$$

$$2 \cdot 5$$

$$3 \cdot 5 = X$$

$$8 \quad 2 \quad \underline{15}$$

$$4 \cdot X = 12$$

$$8 \quad \underline{3} \quad 16$$

9. Which of the numbers in the set (4, 5, 10) should be put in place of the X's in the story to make the story true? (Three points: one point for each number that he openly says will NOT make the story true. If the pupil says, "NONE," he should be given three points.)
10. Which of these expressions (the interviewer should point to each one separately) should be put in place of the X in the story to make the story NOT true? (One point for the answer, $5 + 7$.)
11. Which of these expressions (the interviewer should point to each one separately) should be put in place of the X in the story in order to make the story NOT true? (One point for the answer, $(10 - 4) - 2$.)
12. Which of these expressions (the interviewer should point to each one separately) should be put in place of the X in order to make the story NOT true? (One point for the answer, $2 + 3$.)
13. Which of these expressions (the interviewer should point to each one separately) should be put in place of the X in the story in order to make the story NOT true? (One point for the answer, $(2 \cdot 4) + 1$.)
14. Which of the numbers in the set (8, 2, 15) should be put in place of the X in the story to make the story true? (One point for the answer, 15.)
15. Which of the numbers in the set (8, 3, 16) should be put in place of the X in the story to make the story true? (One point for the answer, 3.)

ALGEBRAIC STRUCTURES

Part II: Solving Problems

(Continued)

Directions for Problems 16, 17, and 18: Give each student a printed number line and say, "This is a number line. You can use it to help you with the next three problems."

Number Line

16. Start at the point 7 - take a trip of 2 - where do you end up on the number line?
(One point for the answer, 9; one point for the answer, 5; total of two points possible.)

Number Line

17. If you end up at the point 4 after a trip of 3, from what point did you start?
(Two points possible: one for the answer, 7; one for the answer, 1.)

Number Line

18. If you start at a point on the number line and go to the right 5 - (pause) - now from this last point go to the left 5. Where do you end up on the number line?
(One point for the answer, "at the same point where I started".)

ALGEBRAIC STRUCTURES
Part III: Applications

Directions: In problems 1, 2, and 3, interviewer points to proper parts of picture as he reads commentary appropriate to that particular problem. On problem 4 the interviewer will set up the apparatus required and perform the indicated demonstration with associated commentary. Check numbered slot at bottom if right.

- Problem 1 - "Henry is wearing a baseball cap (interviewer points to Henry).
Picture 1: John is wearing a space helmet (interviewer points to John). Henry and John each have the same amount of red string (point). Henry and John each have the same amount of green string (point). Each boy has less red string than green string (so indicate)." (Interviewer pushes picture one to one side - still visible to pupil.)
- Problem 1 - "Here we see Henry and John flying their kites. The kites are
Picture 2: all the way out to the end of the string. Is there more string tied to Henry's kite? Or, is there more string tied to John's kite? Or, are the lengths of the strings equal?" (Answer, equal - 1 point.)
- Problem 2 - "Henry and John are going to sharpen their black-and-yellow
Picture 3: pencils. The pencils are the same length. The only difference is that the eraser on Henry's pencil is on the black end, while the eraser on John's pencil is on the yellow end. At the bottom of the picture we see what the pencils look like after they have been sharpened. Whose pencil is the longer, or are they the same length?" (Answer, equal - 1 point.)
- Problem 3 - "John and Henry are shown building a fence. Henry started at
Picture 4: one end and put the short green boards at the bottom and the black boards at the top. John started at the other end and put the long black boards at the bottom and the short green boards at the top. All green boards are the same length and all black boards are the same length. Is Henry's part of the fence higher than John's? Or, is John's part of the fence higher than Henry's? Or, is John's part of the fence the same height as Henry's part of the fence?" (Answer, equal - 1 point.)
- Problem 4 - Interviewer gives student 6 white sticks and places 4 cups on
Apparatus 1: table in front of pupil. "Can you put these 6 white sticks into these 4 cups so that no two cups have the same number of sticks in them?" If student succeeds in distributing white sticks into the cups with no sticks in one cup, 1 stick in one cup, 2 sticks in one cup, and 3 sticks in the last cup, check slot at bottom of page.

Problem Problem Problem Problem

1

2

3

4

ALGEBRAIC STRUCTURES
Part III: Applications

Directions: In problems 5, 6, and 7, interviewer points to proper parts of picture as he reads commentary appropriate to that particular problem. On problem 8 the interviewer will set up the apparatus required and perform the indicated demonstration with associated commentary. Check numbered slot at bottom if right.

- Problem 5 - Henry and John are each going to take an airplane trip from Los Angeles to New York by way of Denver and Chicago. Henry's flight leaves Los Angeles, flies over Denver and lands in Chicago to refuel. It then flies from Chicago to New York. John's flight leaves Los Angeles and lands in Denver for refueling. It then flies over Chicago and on to New York. Does John's plane fly farther than Henry's? Or, does Henry's plane fly farther than John's? Or, do they both fly the same distance? (Answer: equal - 1 point).
- Picture 5:
- Problem 6 - John and Henry are playing with an electric train. John runs the engine around the track and stops when he comes to the red car. He hooks the red car onto the engine and then runs the two of them around the track. He stops by the green car and hooks the green car onto the red car. Later when Henry played with the train, he hooked the red car to the green car and put the two cars on the track. He then ran the engine around the track to where the red and green cars were. He hooked them onto the engine. Was John's train longer than Henry's train? Or was Henry's train longer than John's train? Or, were both trains the same length? (Answer: equal - 1 point.)
- Picture 6:
- Problem 7 - John and Henry were planning to mix some ice cold punch. They have one can of orange juice, one can of lemon juice, and a pitcher full of water. John says they will have more punch if they mix the orange juice and lemon juice FIRST and THEN add the water. Henry says that this is not true. Henry says that they will have more punch if to the lemon juice they first add the water and then add the orange juice to this mixture. Is John right? Or is Henry right? Or would they have the same amount in both cases? (Answer: equal - 1 point.)
- Picture 7,8:
- Problem 8 - Interviewer sets four by four checker board in front of pupil. He points out to him that there are eight black squares and eight red squares. He then detaches the two black squares on opposite corners of the board. He then presents the pupils with the seven rectangular pieces of card board and asks him to try to completely cover the checker board that remains after the two opposite black corner squares have been detached. (There are five (5) points possible on this problem; pupil scores one point if he says that it can't be done; pupil scores two additional points if he notes that seven 2×1 rectangles will cover 14 squares and that there are just 14 squares to be covered; pupil gets full five points if he points out that each rectangular piece of card board will cover 1 black square and 1 red square AND that, therefore, seven 1×2 card boards will cover 7 red and 7 black squares AND that the deleted board contains 6 black and 8 red squares - hence - it cannot be done.)
- Apparatus 2:

ALGEBRAIC STRUCTURES
Part III: Applications

Directions: In problems 9, 10 & 11, interviewer points to proper part of picture as he reads commentary appropriate to that particular problem. On problem 12, the interviewer will set up the apparatus required and perform the indicated demonstration with associated commentary. Check numbered slot at bottom if right.

Problem 9 - John and Henry are playing at the beach. They see two groups of
Picture 9: people. In one group are five boys and five girls. In the other group are five couples. Each couple is made up of one boy and one girl. John says there are more people in the group of couples. Henry says there are more people in the group made up of five boys and five girls. Is John right? Or, is Henry right? Or, are they both wrong? (Answer: both wrong - 1 point.)

Problem 10 - John and Henry visit an Indian Village. The village is divided in-
Picture 10: to two parts by a river. On one side of the river the wigwams are all together and the teepees are all together. On the other side of the river, there is first a wigwam, then a teepee, then another wigwam, and then another teepee. If there are the same number of wigwams in each part of the Indian village AND if there are the same number of teepees in each part of the village, and, further, each wigwam contains two Indians and each teepee contains one Indian, is the number of Indians in one part of the village greater than, equal to, or less than the number of Indians in the other part of the village? (Answer - equal to - 1 point.)

Problem 11 - The doctor tells John and Henry that they must each take three
Picture 11: white pills and three pink pills every day for a week. Henry takes three white pills each morning and three pink pills each night for the five-day week. John takes a white and a pink pill at breakfast, a white and a pink pill at lunch, and a white and a pink pill at supper each day for a five-day week. At the end of the five-day week, does John take more pills, less pills, or the same number of pills as Henry? (Answer: the same number - 1 point.)

Problem 12 - Interviewer sets two glasses in front of pupil, one about half
Apparatus 3: full of clear water and the other about half full of a colored liquid. Interviewer then takes exactly one teaspoonful of the colored liquid over to the clear water and stirs the mixture up well. He then returns a teaspoonful of the mixture to the original glass that contained the colored liquid. He then asks the pupil if he thinks there is more colored liquid in the original glass that contained clear water or if there is more water in the glass that originally contained the colored liquid. (Answer: There is the same - check proper slot at bottom for right answer.)

Problem	Problem	Problem	Problem
9	10	11	12

TEST AR

1		3
---	--	---

(2)

(4)

(5)

APPENDIX II

A Group Arithmetic Test

by

Rodney Skager

2		5
---	--	---

(9)

(6)

(4)

(7)

TEST AR

1		3
---	--	---

2	4	5
---	---	---

2		5
---	--	---

9	6	4	7
---	---	---	---

A R

1		5
---	--	---

⑥

⑦

①

③

4		6
---	--	---

③

②

⑤

⑦

3		5
---	--	---

⑨

④

②

⑥

5		2
---	--	---

④

①

⑥

⑧

6		8
---	--	---

⑨

⑤

⑦

④

0		3
---	--	---

④

①

⑦

⑩

9		5
---	--	---

⑧

③

④

⑨

3		7
---	--	---

⑦

⑧

②

⑥

6		0
---	--	---

⑧

⑩

⑦

⑤

17		19
----	--	----

⑧

⑩

⑱

⑯

TEST NO

3	1		(2)	(1)	(3)	(4)
---	---	--	-----	-----	-----	-----

4	3		(3)	(2)	(4)	(1)
---	---	--	-----	-----	-----	-----

3	3		(1)	(0)	(3)	(2)
---	---	--	-----	-----	-----	-----

4	0		(4)	(1)	(2)	(0)
---	---	--	-----	-----	-----	-----

oo	o		(ooo)	(o)	(oo)	(5)
----	---	--	-------	-----	------	-----

6	1		(4)	(2)	(6)	(5)
---	---	--	-----	-----	-----	-----

6	3		(1)	(3)	(2)	(4)
---	---	--	-----	-----	-----	-----

4	2		(3)	(1)	(2)	(6)
---	---	--	-----	-----	-----	-----

8	3		(5)	(4)	(7)	(6)
---	---	--	-----	-----	-----	-----

10	6		(5)	(11)	(4)	(0)
----	---	--	-----	------	-----	-----

3	1		2	1	3	4
4	3		3	2	4	1
3	3		1	0	3	2
4	0		4	1	2	0
5	2		4	2	3	5
6	1		4	2	6	5
6	3		1	3	2	4
9	5		8	4	7	6
8	3		5	4	7	6
10	6		5	11	4	0

17	10		11	8	10	7
11	1		1	10	12	9
10	6		5	11	4	0
17	15		10	2	4	3
13	3		3	1	11	10
12	9		3	2	4	8
16	7		9	10	6	11
15	8		6	9	8	7
13	7		7	8	6	3
14	6		9	8	4	7

$$\boxed{\frac{3}{3}} \parallel \left(\frac{4}{3} \right) \left(\frac{1}{5} \right) \left(\frac{4}{3} \right) \left(\frac{5}{6} \right)$$

$$\boxed{\frac{2}{2}} \parallel \left(\frac{2}{7} \right) \left(\frac{9}{10} \right) \left(\frac{5}{2} \right) \left(\frac{1}{6} \right)$$

$$\boxed{\frac{6}{1}} \parallel \left(\frac{4}{3} \right) \left(\frac{3}{5} \right) \left(\frac{5}{1} \right) \left(\frac{3}{2} \right)$$

$$\boxed{\frac{3}{1}} \parallel \left(\frac{2}{2} \right) \left(\frac{1}{2} \right) \left(\frac{2}{4} \right) \left(\frac{3}{8} \right)$$

$$\boxed{\frac{5}{3}} \parallel \left(\frac{4}{6} \right) \left(\frac{2}{7} \right) \left(\frac{1}{6} \right) \left(\frac{7}{1} \right)$$

$$\boxed{\frac{3}{7}} \parallel \left(\frac{1}{6} \right) \left(\frac{4}{4} \right) \left(\frac{5}{5} \right) \left(\frac{1}{3} \right)$$

$$\boxed{\frac{2}{3}} \parallel \left(\frac{5}{1} \right) \left(\frac{4}{1} \right) \left(\frac{2}{2} \right) \left(\frac{4}{5} \right)$$

$$\boxed{\frac{2}{9}} \parallel \left(\frac{5}{6} \right) \left(\frac{4}{8} \right) \left(\frac{8}{3} \right) \left(\frac{4}{6} \right)$$

$$\boxed{\frac{7}{3}} \parallel \left(\frac{5}{4} \right) \left(\frac{9}{2} \right) \left(\frac{5}{6} \right) \left(\frac{3}{1} \right)$$

$$\boxed{\frac{6}{6}} \parallel \left(\frac{9}{4} \right) \left(\frac{9}{1} \right) \left(\frac{8}{3} \right) \left(\frac{3}{9} \right)$$

$$\boxed{3 \begin{smallmatrix} / \\ / \end{smallmatrix} 3} \parallel \begin{pmatrix} 4 \\ / \\ / \end{pmatrix} \begin{pmatrix} 1 \\ / \\ / \end{pmatrix} \begin{pmatrix} 4 \\ / \\ / \end{pmatrix} \begin{pmatrix} 5 \\ / \\ / \end{pmatrix}$$

$$\boxed{4 \begin{smallmatrix} / \\ / \end{smallmatrix} 4} \parallel \begin{pmatrix} 2 \\ / \\ / \end{pmatrix} \begin{pmatrix} 9 \\ / \\ / \end{pmatrix} \begin{pmatrix} 6 \\ / \\ / \end{pmatrix} \begin{pmatrix} 1 \\ / \\ / \end{pmatrix}$$

$$\boxed{6 \begin{smallmatrix} / \\ / \end{smallmatrix} 1} \parallel \begin{pmatrix} 4 \\ / \\ / \end{pmatrix} \begin{pmatrix} 3 \\ / \\ / \end{pmatrix} \begin{pmatrix} 5 \\ / \\ / \end{pmatrix} \begin{pmatrix} 3 \\ / \\ / \end{pmatrix}$$

$$\boxed{5 \begin{smallmatrix} / \\ / \end{smallmatrix} 5} \parallel \begin{pmatrix} 6 \\ / \\ / \end{pmatrix} \begin{pmatrix} 1 \\ / \\ / \end{pmatrix} \begin{pmatrix} 7 \\ / \\ / \end{pmatrix} \begin{pmatrix} 3 \\ / \\ / \end{pmatrix}$$

$$\boxed{5 \begin{smallmatrix} / \\ / \end{smallmatrix} 3} \parallel \begin{pmatrix} 4 \\ / \\ / \end{pmatrix} \begin{pmatrix} 2 \\ / \\ / \end{pmatrix} \begin{pmatrix} 1 \\ / \\ / \end{pmatrix} \begin{pmatrix} 7 \\ / \\ / \end{pmatrix}$$

$$\boxed{2 \begin{smallmatrix} / \\ / \end{smallmatrix} 7} \parallel \begin{pmatrix} 1 \\ / \\ / \end{pmatrix} \begin{pmatrix} 4 \\ / \\ / \end{pmatrix} \begin{pmatrix} 5 \\ / \\ / \end{pmatrix} \begin{pmatrix} 1 \\ / \\ / \end{pmatrix}$$

$$\boxed{4 \begin{smallmatrix} / \\ / \end{smallmatrix} 5} \parallel \begin{pmatrix} 2 \\ / \\ / \end{pmatrix} \begin{pmatrix} 6 \\ / \\ / \end{pmatrix} \begin{pmatrix} 10 \\ / \\ / \end{pmatrix} \begin{pmatrix} 3 \\ / \\ / \end{pmatrix}$$

$$\boxed{2 \begin{smallmatrix} / \\ / \end{smallmatrix} 9} \parallel \begin{pmatrix} 6 \\ / \\ / \end{pmatrix} \begin{pmatrix} 4 \\ / \\ / \end{pmatrix} \begin{pmatrix} 8 \\ / \\ / \end{pmatrix} \begin{pmatrix} 4 \\ / \\ / \end{pmatrix}$$

$$\boxed{7 \begin{smallmatrix} / \\ / \end{smallmatrix} 3} \parallel \begin{pmatrix} 6 \\ / \\ / \end{pmatrix} \begin{pmatrix} 9 \\ / \\ / \end{pmatrix} \begin{pmatrix} 5 \\ / \\ / \end{pmatrix} \begin{pmatrix} 8 \\ / \\ / \end{pmatrix}$$

$$\boxed{6 \begin{smallmatrix} / \\ / \end{smallmatrix} 6} \parallel \begin{pmatrix} 9 \\ / \\ / \end{pmatrix} \begin{pmatrix} 10 \\ / \\ / \end{pmatrix} \begin{pmatrix} 8 \\ / \\ / \end{pmatrix} \begin{pmatrix} 3 \\ / \\ / \end{pmatrix}$$

EO-2

$\boxed{7/7}$	$\textcircled{3/12}$	$\textcircled{9/6}$	$\textcircled{10/4}$	$\textcircled{4/9}$
$\boxed{4/9}$	$\textcircled{8/5}$	$\textcircled{2/12}$	$\textcircled{6/6}$	$\textcircled{6/8}$
$\boxed{10/3}$	$\textcircled{9/5}$	$\textcircled{2/12}$	$\textcircled{3/9}$	$\textcircled{7/6}$
$\boxed{4/8}$	$\textcircled{7/5}$	$\textcircled{2/9}$	$\textcircled{5/8}$	$\textcircled{6/5}$
$\boxed{6/8}$	$\textcircled{6/7}$	$\textcircled{9/6}$	$\textcircled{5/9}$	$\textcircled{8/7}$
$\boxed{8/8}$	$\textcircled{5/9}$	$\textcircled{9/6}$	$\textcircled{7/10}$	$\textcircled{10/6}$
$\boxed{5/10}$	$\textcircled{7/7}$	$\textcircled{9/6}$	$\textcircled{7/9}$	$\textcircled{6/8}$
$\boxed{11/4}$	$\textcircled{8/7}$	$\textcircled{8/6}$	$\textcircled{8/9}$	$\textcircled{7/5}$
$\boxed{2/14}$	$\textcircled{10/7}$	$\textcircled{5/11}$	$\textcircled{9/8}$	$\textcircled{6/9}$
$\boxed{9/9}$	$\textcircled{12/5}$	$\textcircled{10/7}$	$\textcircled{7/11}$	$\textcircled{10/5}$

APPENDIX III

The Group Transfer Test

[Note: Each child was given a booklet in which one problem appeared per page. Adjacent pages were always of different colors. Children were not permitted to turn the page until told to do so. Pages from the booklet have been inserted into the instructions at the appropriate point to exhibit what the child was shown in his booklet. In the upper left-hand corner for each problem he indicated the problem number and the color of the page on which this problem was presented. This information did not appear on the child's booklet. The administrator of the test followed the instructions indicated below.]

Instructions

Write on board:

$$2 + 3 = 5$$

5
3
2

Say: "Here on the board it says $2 + 3 = 5$. I see 2 and 3 make 5. Now these numbers in the board are not in the right order. If I put 3 in place of the 2, it will make a story which is true. Now I put 5 in place of this 5."

APPENDIX III**The Group Transfer Test**

Draw 2 and put 15 = Five.

by

Evan R. Keislar

"Can I get it? No, 2 and 3 don't equal 5. 2 plus 3 doesn't equal 5. That isn't a story. Can I put 1 instead of 3? No, 2 plus 1 doesn't equal 5. The board is certainly true. Now we say something that you know is right: $2 + 3 = 5$. This is a true story. It is right. To show that 5 is the right number, let's take a line through it. Like this?"

Repeat with

$$3 + 2 = 5$$

5
2
3

Then say: "Now look at the first page of your booklet. It is white."

Write
Problem 1

$$3 + 3 = 2$$

3
3
2

APPENDIX III

The Group Transfer Test.

(Note: Each child was given a booklet in which one problem appeared per page. Adjacent pages were always of different colors. Children were not permitted to turn the page until told to do so. Pages from the booklet have been inserted into the instructions at the appropriate point to exhibit what the child was shown in his booklet. In the upper left-hand corner for each problem is indicated the problem number and the color of the page on which this problem was presented. This information did not appear on the child's booklet. The administrator of the test followed the instructions indicated below.)

Instructions

Write on board:

$$2 + 4 = X$$

5

7

6

Say: "Here on the board it says $2 + 4 = X$ or 2 and 4 make X. See these numbers in the boxes over here. If you put one of these numbers in place of the X, it will say something right; it will make a story which is true. Now what number here should I put in place of this X?"

Erase X and put in a five.

"Can I put 5? No. 2 and 4 don't make 5. 2 plus 4 doesn't equal 5. That isn't a story which is true. Can I put 7 instead of X? No. 2 plus 4 doesn't equal 7. How about 6? Certainly. Now we say something that you know is right: $2 + 4 = 6$. This is a true story. It is right. To show that 6 is the right number, let's draw a line through it. Like this."

Repeat with

$$5 + X = 8$$

2

3

4

Then say: "Now look at the first page of your booklet. It is white."

White
Problem 1

$$3 + 5 = X$$

7

8

9

"Here it says that $3 + 5 = X$. What number in the boxes at the side is the one which should go in place of X ? What number will make a true story if you put it down instead of X ? What number will make this say something right? Find the number and draw a line through it. Do it now."

Yellow

Problem 2

$4 + X = 7$	1
	5
	3

"Now turn the page and you will see a yellow page. Does everyone have a yellow page? Now what does the story on this page say? It says $4 + X = 7$. What number should you put in place of X to make this story true; to make it say something which is right? Draw a line through the right number."

Pink

Problem 3

$X + 5 = 9$	4
	6
	2

"Now turn the page and you will find a pink page. Now the story on this page says: $X + 5 = 9$. What number in the box is the right number, the one that will make this story true if it takes the place of X ? Draw a line through the right number."

White

Problem 4

$7 - 4 = X$	3
	11
	6

"Now turn the page and you will find a white page. Now the story on this page says: $7 - 4 = X$. What number in the box is the right number, the one that will make this story true if it takes the place of X ? Draw a line through the right number."

Yellow

Problem 5

$10 - X = 6$	2
	4
	5

"Now turn the page and you will find a yellow page. Now the story on this page says: $10 - X = 6$. What number in the box is the right number, the one that will make this story true if it takes the place of X ? Draw a line through the right number."

Pink

Problem 6

	8
$X - 7 = 2$	9
	10

"Now turn the page and you will find a pink page. Now the story on this page says: $X - 7 = 2$. What number in the box is the right number, the one that will make this story true if it takes the place of X? Draw a line through the right number."

Do not have them turn to the next page yet.

Write on the board:

$$6 > 4.$$

Say: "Here is a story that says 6 is bigger than 4. See this sign means bigger or larger than. 6 is bigger than 4. Is this true? Yes this is a true story. 6 really is bigger than 4."

"Now here is another story."

Write on board:

$$2 < 3.$$

"It says that two is smaller than three. And it is a true story too."

White

Problem 7

	4
$X > 5$	3
	3

"Now look at the next page in your booklets, the white page. This story says that X is bigger than 5. What number in the box should you put in place of X to make this story true? Draw a line through that number. This story says that X is bigger or larger than 5."

Yellow

Problem 8

	5
$7 < X$	7
	9

"This story says that 7 is smaller than X. What number in the box should be put in place of X to make this story true? Draw a line

through that number. This story says that X is bigger or larger than 5.

Pink

Problem 9

		2
	$k > X$	4
		8

"Turn to the pink page. This story says that k is bigger than X . What number in the box should be put in place of X to make this story true? Draw a line through this number."

Blue

T	T	☐
H	H	☐
U	U	☐

"Here are some letters. The letter at the top is a T. There is a T drawn again inside a box. Make another T in the empty box. Do this now. Here is an H. Make an H inside the empty box. Here is a U. Make a U inside the empty box."

White

$3 + 4 = 7$
$\boxed{+ \quad =}$

"Here is a story which says that $3 + 4 = 7$. Inside the box is part of another story but the numbers are missing. Now don't write anything. Let me ask you a question. Can we finish the story inside the box so that it says the same thing as the story on top but in a different way? Yes we can."

Yellow

$3 + 4 = 7$
$\boxed{4 + 3 = 7}$

"Turn to the yellow page and see. Here inside the box is the finished story. Do you see that it says the same thing as the story on top but in a different way. The finished story says $4 + 3 = 7$. The story on top says $3 + 4 = 7$."

Pink

Problem 10

$T + U = H$	
<table border="1" style="width: 50%; margin: auto;"> <tr> <td>$+ =$</td> </tr> </table>	$+ =$
$+ =$	

"Now turn to the pink page. Here is a story that says that something called T plus something called U equals something called H. It says that T plus U equals H.

"There is an unfinished story inside the box. Write in letters to finish this story so that it says the same thing as the story on top but in a different way. Do it now with your pencil."

White

$5 + 1 = 6$	
<table border="1" style="width: 50%; margin: auto;"> <tr> <td>$= +$</td> </tr> </table>	$= +$
$= +$	

"Now here is a story which says $5 + 1 = 6$. There is an unfinished story in the box. Can we finish this story so that it says the same thing as the story on top but in a different way. Yes we can.

Yellow

$5 + 1 = 6$	
<table border="1" style="width: 50%; margin: auto;"> <tr> <td>$6 = 5 + 1$</td> </tr> </table>	$6 = 5 + 1$
$6 = 5 + 1$	

"Turn to the yellow page and see. Here on the yellow page in the finished story. It says $6 = 5 + 1$. This story in the box says the same thing as the story on top but in a different way.

Pink

Problem 11

$T + U = H$	
<table border="1" style="width: 50%; margin: auto;"> <tr> <td>$= +$</td> </tr> </table>	$= +$
$= +$	

"Now look at the next pink page. This story on top says that T plus U = H. There is an unfinished story in the box. Write in the letters to finish this story so that it says the same thing as the story on top but in a different way. Do it now with your pencil."

White

$$2 + 4 = 6$$

$$= -$$

"Here is a story which says $2 + 4 = 6$. There is an unfinished story in the box. Can we finish this story so that it says the same thing as the story on top but in a different way? Yes we can."

Yellow

$$2 + 4 = 6$$

$$4 = 6 - 2$$

"Turn to the yellow page and see. Here in the yellow page is the finished story. It says $4 = 6 - 2$. This story in the box says the same thing as the story on top but in a different way."

Pink

Problem 12

$$T + U = H$$

$$= -$$

"Now turn to the pink page. This story on top says that T plus $U = H$. There is an unfinished story in the box. Write in the letters to finish this story so that it says the same thing as the story on top but in a different way. Do it now with your pencil."

Blue

Problem 13

$$H = T + U$$

$$= +$$

"Now turn to the blue page. This story on top says $H = T + U$. There is an unfinished story in the box. Write in the letters to finish this story so that it says the same thing as the story on top but in a different way. Do it now with your pencil."

White

Problem 14

$T - U = E$	
<table border="1" style="width: 50%; margin: auto;"> <tr> <td>$-$</td> </tr> </table>	$-$
$-$	

"Now turn to the white page. This story on top says T minus or T take away U = E. There is an unfinished story in the box. Write in the letters to finish this story so that it says the same thing as the story on top but in a different way. Do it now with your pencil."

Blue

Problem 15

$3 + 4 = X$	8
	9
	10

"Now turn the page and you will find a blue page. Now the story on this page says: $3 + 4 = X$. What number in the box is the right number, the one that will make this story true if it takes the place of X? Draw a line through the right number."

White

Problem 16

$X - 6 = 2$	5
	4
	6

"Now turn the page and you will find a white page. Now the story on this page says: $X - 6 = 2$. What number in the box is the right number, the one that will make this story true if it takes the place of X? Draw a line through the right number."